10. Graph Matrices

Since a graph is completely determined by specifying either its adjacency structure or its incidence structure, these specifications provide far more efficient ways of representing a large or complicated graph than a pictorial representation. As computers are more adept at manipulating numbers than at recognising pictures, it is standard practice to communicate the specification of a graph to a computer in matrix form. In this chapter, we study various types of matrices associated with a graph, and our study is based on Narsing Deo [63], Foulds [82], Harary [104] and Parthasarathy [180].

10.1 Incidence Matrix

Let G be a graph with n vertices, m edges and without self-loops. The incidence matrix A of G is an $n \times m$ matrix $A = [a_{ij}]$ whose n rows correspond to the n vertices and the m columns correspond to m edges such that

$$a_{ij} = \begin{cases} 1, & \textit{if jth edge } m_j \textit{ is incident on the ith vertex} \\ 0, & \textit{otherwise.} \end{cases}$$

It is also called *vertex-edge incidence matrix* and is denoted by $A(G)$.

Example Consider the graphs given in Figure 10.1. The incidence matrix of G_1 is

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The incidence matrix of G_2 is

$$
A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.
$$

The incidence matrix of G_3 is

$$
A(G_3) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}
\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.
$$

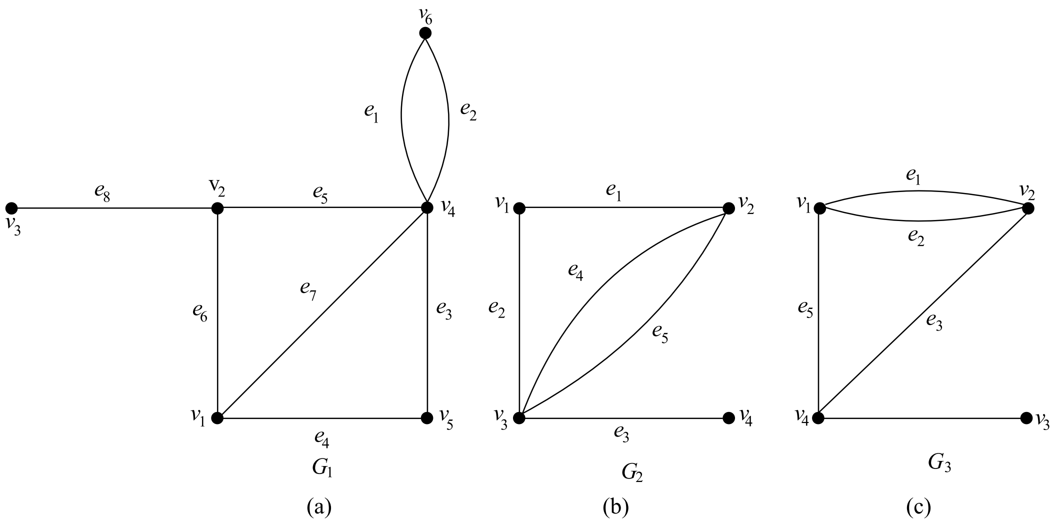

Fig. 10.1

The incidence matrix contains only two types of elements, 0 and 1. This clearly is a binary matrix or a (0, 1)-matrix.

We have the following observations about the incidence matrix A.

1. Since every edge is incident on exactly two vertices, each column of A has exactly two one's.

2. The number of one's in each row equals the degree of the corresponding vertex.

3. A row with all zeros represents an isolated vertex.

4. Parallel edges in a graph produce identical columns in its incidence matrix.

5. If a graph is disconnected and consists of two components G_1 and G_2, the incidence matrix $A(G)$ of graph G can be written in a block diagonal form as

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix},$$

where $A(G_1)$ and $A(G_2)$ are the incidence matrices of components G_1 and G_2. This observation results from the fact that no edge in G_1 is incident on vertices of G_2 and vice versa. Obviously, this is also true for a disconnected graph with any number of components.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

Note The matrix A has been defined over a field, Galois field modulo 2 or $GF(2)$, that is, the set $\{0,1\}$ with operation addition modulo 2 written as + such that $0+0=0,\ 1+0=1,\ 1+1=0$ and multiplication modulo 2 written as "." such that $0.0=0,\ 1.0=0=0.1,\ 1.1=1$.

The following result is an immediate consequence of the above observations.

Theorem 10.1 Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutation of rows and columns.

Proof Let the graphs G_1 and G_2 be isomorphic. Then there is a one-one correspondence between the vertices and edges in G_1 and G_2 such that the incidence relation is preserved. Thus $A(G_1)$ and $A(G_2)$ are either same or differ only by permutation of rows and columns.

The converse follows, since permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph. ❑

Rank of the incidence matrix

Let G be a graph and let $A(G)$ be its incidence matrix. Now each row in $A(G)$ is a vector over $GF(2)$ in the vector space of graph G. Let the row vectors be denoted by $A_1, A_2, \ldots, A_n$. Then,

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix}.$$

Since there are exactly two ones in every column of A, the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries).

Thus vectors $A_1, A_2, \ldots, A_n$ are linearly dependent. Therefore, rank $A < n$.

Hence, rank $A \leq n-1$.

From the above observations, we have the following result.

Theorem 10.2 If $A(G)$ is an incidence matrix of a connected graph G with n vertices, then rank of $A(G)$ is $n-1$.

Proof Let G be a connected graph with n vertices and let the number of edges in G be m. Let $A(G)$ be the incidence matrix and let $A_1, A_2, \ldots, A_n$ be the row vector of $A(G)$.

$$\text{Then, } A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix}. \tag{10.2.1}$$

Clearly, rank $A(G) \leq n-1$. (10.2.2)

Consider the sum of any m of these row vectors, $m \leq n-1$. Since G is connected, $A(G)$ cannot be partitioned in the form

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

such that $A(G_1)$ has m rows and $A(G_2)$ has $n-m$ rows.

Thus there exists no $m \times m$ submatrix of $A(G)$ for $m \leq n-1$, such that the modulo 2 sum of these m rows is equal to zero.

As there are only two elements 0 and 1 in this field, the additions of all vectors taken m at a time for $m = 1, 2, \ldots, n-1$ gives all possible linear combinations of $n-1$ row vectors.

Thus no linear combinations of m row vectors of A, for $m \leq n-1$, is zero.

Therefore, rank $A(G) \leq n-1$. (10.2.3)

Combining (10.2.2) and (10.2.3), it follows that rank $A(G) = n-1$. ❑

Remark If G is a disconnected graph with k components, then it follows from the above theorem that rank of $A(G)$ is $n-k$.

Let G be a connected graph with n vertices and m edges. Then the order of the incidence matrix $A(G)$ is $n \times m$. Now, if we remove any one row from $A(G)$, the remaining $(n-1)$ by m submatrix is of rank $(n-1)$. Thus the remaining $(n-1)$ row vectors are linearly independent. This shows that only $(n-1)$ rows of an incidence matrix are required to specify the corresponding graph completely, because $(n-1)$ rows contain the same information as the entire matrix. This follows from the fact that given $(n-1)$ rows, we can construct the nth row, as each column in the matrix has exactly two ones. Such an $(n-1) \times m$ matrix of A is called a *reduced incidence matrix* and is denoted by A_f. The vertex corresponding to the deleted row in A_f is called the *reference vertex*. Obviously, any vertex of a connected graph can be treated as the reference vertex.

The following result gives the nature of the incidence matrix of a tree.

Theorem 10.3 The reduced incidence matrix of a tree is non-singular.

Proof A tree with n vertices has $n-1$ edges and also a tree is connected. Therefore, the reduced incidence matrix is a square matrix of order $n-1$, with rank $n-1$. Thus the result follows.

Now a graph G with n vertices and $n-1$ edges which is not a tree is obviously disconnected. Therefore the rank of the incidence matrix of G is less than $(n-1)$. Hence the $(n-1)\times(n-1)$ reduced incidence matrix of a graph is non-singular if and only if the graph is a tree. ❑

10.2 Submatrices of A(G)

Let H be a subgraph of a graph G, and let $A(H)$ and $A(G)$ be the incidence matrices of H and G respectively. Clearly, $A(H)$ is a submatrix of $A(G)$, possibly with rows or columns permuted. We observe that there is a one-one correspondence between each $n\times k$ submatrix of $A(G)$ and a subgraph of G with k edges, k being a positive integer, $k < m$ and n being the number of vertices in G.

The following is a property of the submatrices of $A(G)$.

Theorem 10.4 Let $A(G)$ be the incidence matrix of a connected graph G with n vertices. An $(n-1)\times(n-1)$ submatrix of $A(G)$ is non-singular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitutes a spanning tree in G.

Proof Let G be a connected graph with n vertices and m edges. So, $m \geq n-1$.

Let $A(G)$ be the incidence matrix of G, so that $A(G)$ has n rows and m columns $(m \geq n-1)$.

We know every square submatrix of order $(n-1)\times(n-1)$ in $A(G)$ is the reduced incidence matrix of some subgraph H in G with $n-1$ edges, and vice versa. We also know that a square submatrix of $A(G)$ is non-singular if and only if the corresponding subgraph is a tree.

Obviously, the tree is a spanning tree because it contains $n-1$ edges of the n-vertex graph.

Hence $(n-1)\times(n-1)$ submatrix of $A(G)$ is non-singular if and only if $n-1$ edges corresponding to $n-1$ columns of this matrix forms a spanning tree. ❑

The following is another form of incidence matrix.

Definition: The matrix $F = [f_{ij}]$ of the graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, is the $n \times m$ matrix associated with a chosen orientation of the edges of G in which for each $e = (v_i, v_j)$, one of v_i or v_j is taken as positive end and the other as negative end, and is defined by

$$f_{ij} = \begin{cases} 1, & if\ v_i\ is\ the\ positive\ end\ of\ e_j, \\ -1, & if\ v_i\ is\ the\ negative\ end\ of\ e_j, \\ 0, & if\ v_i\ is\ not\ incident\ with\ e_j. \end{cases}$$

This matrix F can also be obtained from the incidence matrix A by changing either of the two 1s to -1 in each column.

The above arguments amount to arbitrarily orienting the edges of G, and F is then the incidence matrix of the oriented graph.

The matrix F is then the modified definition of the incidence matrix A.

Example Consider the graph G shown in Figure 10.2, with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$.

The incidence matrix is given by

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Therefore,

$$F = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

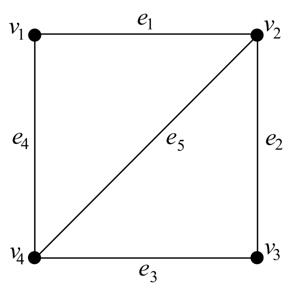

Fig. 10.2

Theorem 10.5 If G is a connected graph with n vertices, then rank $F = n-1$.

Proof Let G be a connected graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Then the matrix $F = [f_{ij}]_{n\times m}$ is given by

$$f_{ij} = \begin{cases} 1, & if\ v_i\ is\ the\ positive\ end\ of\ e_j, \\ -1, & if\ v_i\ is\ the\ negative\ end\ of\ e_j, \\ 0, & if\ v_i\ is\ not\ incident\ with\ e_j. \end{cases}$$

Let R_j be the jth row of F. Since each column of F has only one $+1$ and one -1, as non-zero entries, rank $F < n$. Thus, rank $F \leq n-1$.

Now, let $\sum_1^n c_j R_j = 0$ be any other linear dependence relation of $R_1, R_2, \ldots, R_n$ with at least one c_j non-zero.

If $c_r \neq 0$, then the row R_r has non-zero entries in those columns which correspond to edges incident with v_r. For each such column there is just one row, say R_s, at which there is a non-zero entry (with opposite sign to the non-zero entry in R_r). The dependence relation thus requires $c_s = c_r$, for all s corresponding to vertices adjacent to v_r. Since G is connected, we have $c_j = c$, for all $j = 1, 2, \ldots, n$. Therefore the dependence relation is $c\left(\sum_1^n R_j\right) = 0$, which is same as the first one. Hence, rank $F = n-1$.

Alternative Proof

Then

$$F = \begin{bmatrix} R_1 \\ R_2 \\ M \\ R_n \end{bmatrix}. \tag{10.5.1}$$

Since each column of F has only one +1 and one -1 as non-zero entries, $\sum_{j=1}^n R_j = 0$. Thus, rank $F \leq n-1$. (10.5.2)

Consider the sum of any m of these row vectors, $m \leq n-1$. As G is connected, F cannot be partitioned in the form $F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$, such that F_1 has m rows and F_2 as $n-m$ rows. Therefore there exists no $m \times m$ submatrix of F for $m \leq n-1$, such that the sum of these m rows is equal to zero. Therefore,

$$\sum_{j=1}^m R_j \neq 0. \tag{10.5.3}$$

Also, there is no linear combination of $m(m \leq n-1)$ vectors of F, which is zero. For, if $\sum_{j=1}^{m} c_j R_j = 0$ is a linear combination, with at least one c_j non-zero, say $c_r \neq 0$, then the row R_r has non-zero entries in those columns which correspond to edges incident with v_r. So for each such column there is just one row, say R_s at which there is a non-zero entry with opposite sign to the non-zero entry in R_r. The linear combination thus requires $c_s = c_r$ for all s corresponding to vertices adjacent to v_r. As G is connected, we have $c_j = c$, for all $j = 1, \ldots, m$. Therefore the linear combination becomes $c\left(\sum_{j=1}^{m} R_j\right) = 0$, or $\sum_{j=1}^{m} R_j = 0$, which contradicts (10.5.3).

Hence, rank $F = n-1$. ❑

Theorem 10.6 If G is a disconnected graph with k components, then rank $F = n-k$.

Proof Since G has k components, F can be partitioned as

$$F = \begin{bmatrix} F_1 & 0 & 0 & \cdots & 0 \\ 0 & F_2 & 0 & \cdots & 0 \\ 0 & 0 & F_3 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & F_k \end{bmatrix},$$

where F_i is the matrix of the ith component G_i of G. We have proved that rank $F_i = n_i - 1$, where n_i is the number of vertices in G_i.

Thus, rank $F = n_1 - 1 + n_2 - 1 + \ldots + n_k - 1 = n_1 + n_2 + \ldots + n_k - k = n - k$,

as the number of vertices in G is $n_1 + n_2 + \ldots + n_k = n$. ❑

Corollary 10.1 A basis for the row space of F is obtained by taking for each i, $1 \leq i \leq k$, any $n_i - 1$ rows of F_i.

Theorem 10.7 The determinant of any square submatrix of the matrix F of a graph G has value 1, -1, or zero.

Proof Let N be the square submatrix of F such that N has both non-zero entries $+1$ and -1 in each column. Then row sum of N is zero and hence $|N| = 0$. Clearly if N has no non-zero entries, then $|N| = 0$.

Now let some column of N have only one non-zero entry. Then expanding $|N|$ with the help of this column, we get $|N| = \pm|N'|$, where N' is a matrix obtained by omitting a row and column of N. Continuing in this way, we either get a matrix whose determinant is zero, or end up with a single non-zero entry of N, in which case $|N| = \pm 1$. ❑

Theorem 10.8 Let X be any set of $n-1$ edges of the connected graph $G=(V, E)$ and F_x the $(n-1)\times(n-1)$ submatrix of the matrix F of G, determined by any $n-1$ rows and those columns which correspond to the edges of X. Then F_x is non-singular if and only if the edge induced subgraph $< X >$ of G is a spanning tree of G.

Proof Let F' be the matrix corresponding to $< X >$. If $< X >$ is a spanning tree of G, then F_x consists of $n-1$ rows of F'. Since $< X >$ is connected, therefore rank $F_x = n-1$. Hence F_x is non-singular.

Conversely, let F_x be non-singular. Then F' contains an $(n-1)\times(n-1)$ non singular submatrix. Therefore, rank $F' = n-1$. Since rank + nullity $= m$, for any graph G, and $m(< X >) = n-1$ and rank $(< X >) = n-1$, therefore, nullity $(< X >) = 0$. Thus $< X >$ is acyclic and connected and so is a spanning tree of G. ❑

10.3 Cycle Matrix

Let the graph G have m edges and let q be the number of different cycles in G. The cycle matrix $B = [b_{ij}]_{q\times m}$ of G is a $(0, 1)-$ matrix of order $q\times m$, with $b_{ij}=1$, if the ith cycle includes jth edge and $b_{ij}=0$, otherwise. The cycle matrix B of a graph G is denoted by $B(G)$.

Example Consider the graph G_1 given in Figure 10.3.

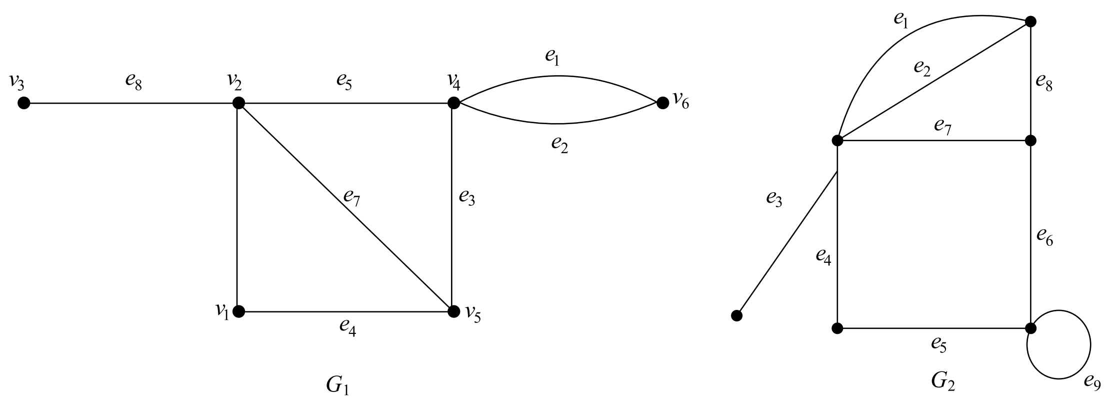

Fig. 10.3

The graph G_1 has four different cycles $Z_1=\{e_1, e_2\}$, $Z_2=\{e_3, e_5, e_7\}$, $Z_3=\{e_4, e_6, e_7\}$ and $Z_4=\{e_3, e_4, e_6, e_5\}$.

The cycle matrix is

$$B(G_1) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

The graph G_2 of Figure 10.3 has seven different cycles, namely, $Z_1 = \{e_1, e_2\}$, $Z_2 = \{e_2, e_7, e_8\}$, $Z_3 = \{e_1, e_7, e_8\}$, $Z_4 = \{e_4, e_5, e_6, e_7\}$, $Z_5 = \{e_2, e_4, e_5, e_6, e_8\}$, $Z_6 = \{e_1, e_4, e_5, e_6, e_8\}$ and $Z_7 = \{e_9\}$. The cycle matrix is given by

$$B(G_2) = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

We have the following observations regarding the cycle matrix $B(G)$ of a graph G.

1. A column of all zeros corresponds to a non cycle edge, that is, an edge which does not belong to any cycle.

2. Each row of $B(G)$ is a cycle vector.

3. A cycle matrix has the property of representing a self-loop and the corresponding row has a single one.

4. The number of ones in a row is equal to the number of edges in the corresponding cycle.

5. If the graph G is separable (or disconnected) and consists of two blocks (or components) H_1 and H_2, then the cycle matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \begin{bmatrix} B(H_1) & 0 \\ 0 & B(H_2) \end{bmatrix},$$

   where $B(H_1)$ and $B(H_2)$ are the cycle matrices of H_1 and H_2. This follows from the fact that cycles in H_1 have no edges belonging to H_2 and vice versa.

6. Permutation of any two rows or columns in a cycle matrix corresponds to relabeling the cycles and the edges.

7. We know two graphs G_1 and G_2 are 2-isomorphic if and only if they have cycle correspondence. Thus two graphs G_1 and G_2 have the same cycle matrix if and only if G_1 and G_2 are 2-isomorphic. This implies that the cycle matrix does not specify a graph completely, but only specifies the graph within 2-isomorphism.

For example, the two graphs given in Figure 10.4 have the same cycle matrix. They are 2-isomorphic, but are not isomorphic.

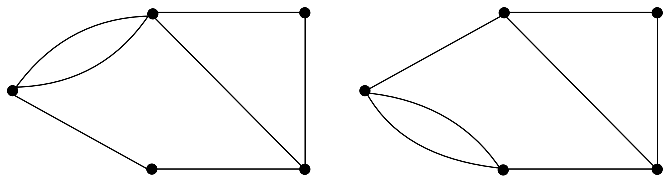

Fig. 10.4

The following result relates the incidence and cycle matrix of a graph without self-loops.

Theorem 10.9 If G is a graph without self-loops, with incidence matrix A and cycle matrix B whose columns are arranged using the same order of edges, then every row of B is orthogonal to every row of A, that is $AB^T = BA^T \equiv 0$ (mod2), where A^T and B^T are the transposes of A and B respectively.

Proof Let G be a graph without self-loops, and let A and B, respectively, be the incidence and cycle matrix of G.

We know that in G for any vertex v_i and for any cycle Z_j, either $v_i \in Z_j$ or $v_i \notin Z_j$.

In case $v_i \notin Z_j$, then there is no edge of Z_j which is incident on v_i and if $v_i \in Z_j$, then there are exactly two edges of Z_j which are incident on v_i.

Now, consider the ith row of A and the jth row of B (which is the jth column of B^T).

Since the edges are arranged in the same order, the rth entries in these two rows are both non-zero if and only if the edge e_r is incident on the ith vertex v_i and is also in the jth cycle Z_j.

We have $[AB^T]_{ij} = \sum [A]_{ir}[B^T]_{rj} = \sum [A]_{ir}[B]_{jr} = \sum a_{ir}b_{jr}$.

For each e_r of G, we have one of the following cases.

i. e_r is incident on v_i and $e_r \notin Z_j$. Here $a_{ir} = 1$, $b_{jr} = 0$.

ii. e_r is not incident on v_i and $e_r \in Z_j$. In this case, $a_{ir} = 0$, $b_{jr} = 1$.

iii. e_r is not incident on v_i and $e_r \notin Z_j$, so that $a_{ir} = 0$, $b_{jr} = 0$.

All these cases imply that the ith vertex v_i is not in the jth cycle Z_j and we have $[AB^T]_{ij} = 0 \equiv 0$ (mod2).

iv. e_r is incident on v_i and $e_r \in Z_j$.

Here we have exactly two edges, say e_r and e_t incident on v_i so that $a_{ir} = 1$, $a_{it} = 1$, $b_{jr} = 1$, $b_{jt} = 1$. Therefore, $[AB^T]_{ij} = \sum a_{ir} b_{jr} = 1 + 1 \equiv 0$ (mod2). ❑

We illustrate the above theorem with the following example (Fig. 10.5).

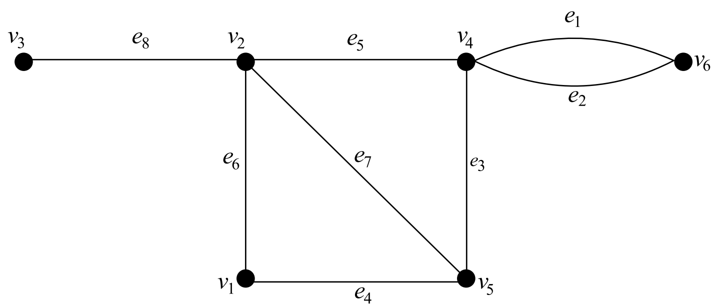

Fig. 10.5

Clearly,

$$AB^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \equiv 0(\mathrm{mod}2).$$

We know that a set of fundamental cycles (or basic cycles) with respect to any spanning tree in a connected graph are the only independent cycles in a graph. The remaining cycles can be obtained as ring sums (i.e., linear combinations) of these cycles. Thus, in a cycle matrix, if we take only those rows that correspond to a set of fundamental cycles and remove all other rows, we do not lose any information. The removed rows can be formed from the rows corresponding to the set of fundamental cycles. For example, in the cycle matrix of the graph given in Figure 10.6, the fourth row is simply the mod 2 sum of the second and the third rows. Fundamental cycles are

$Z_1 = \{e_1,\ e_2,\ e_4,\ e_7\}$
$Z_2 = \{e_3,\ e_4,\ e_7\}$
$Z_3 = \{e_5,\ e_6,\ e_7\}$

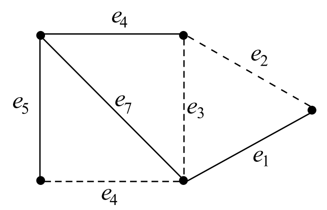

Fig. 10.6

$$\begin{array}{c} \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_2 & e_3 & e_6 & & e_1 & e_4 & e_5 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

A submatrix of a cycle matrix in which all rows correspond to a set of fundamental cycles is called a *fundamental cycle matrix* B_f.

The permutation of rows and/or columns do not affect B_f. If n is the number of vertices, m the number of edges in a connected graph G, then B_f is an $(m-n+1)\times m$ matrix because the number of fundamental cycles is $m-n+1$, each fundamental cycle being produced by one chord.

Now, arranging the columns in B_f such that all the $m-n+1$ chords correspond to the first $m-n+1$ columns and rearranging the rows such that the first row corresponds to the fundamental cycle made by the chord in the first column, the second row to the fundamental cycle made by the second, and so on. This arrangement is done for the above fundamental cycle matrix.

A matrix B_f thus arranged has the form

$$B_f = [I_\mu : B_t],$$

where I_μ is an identity matrix of order $\mu = m-n+1$ and B_t is the remaining $\mu \times (n-1)$ submatrix, corresponding to the branches of the spanning tree.

From equation $B_f = [I_\mu : B_t]$, we have rank $B_f = \mu = m-n+1$.

Since B_f is a submatrix of the cycle matrix B, therefore, rank $B \geq$ rank B_f and thus,

$$\text{rank } B \geq m-n+1.$$

The following result gives the rank of the cycle matrix.

Theorem 10.10 If B is a cycle matrix of a connected graph G with n vertices and m edges, then rank $B = m-n+1$.

Proof Let A be the incidence matrix of the connected graph G.

Then $AB^T \equiv 0(\text{mod}2)$.

Using Sylvester's theorem (Theorem 10.13), we have rank $A+$ rank $B^T \le m$ so that rank $A+$ rank $B \le m$.

Therefore, rank $B \le m-$ rank A.

As rank $A = n-1$, we get rank $B \le m-(n-1) = m-n+1$.

But, rank $B \ge m-n+1$.

Combining, we get rank $B = m-n+1$. ❑

Theorem 10.10 can be generalised in the following form.

Theorem 10.11 If B is a cycle matrix of a disconnected graph G with n vertices, m edges and k components, then rank $B = m-n+k$.

Proof Let B be the cycle matrix of the disconnected graph G with n vertices, m edges and k components. Let the k components be G_1, $G_2, \ldots, G_k$ with $n_1, n_2, \ldots, n_k$ vertices and m_1, $m_2, \ldots, m_k$ edges respectively.

Then $n_1+n_2+\ldots+n_k = n$ and $m_1+m_2+\ldots+m_k = m$.

Let B_1, B_2, …, B_k be the cycle matrices of G_1, G_2, …, G_k.

$$\text{Then } B(G) = \begin{bmatrix} B_1(G_1) & 0 & 0 & \cdots & 0 \\ 0 & B_2(G_2) & 0 & \cdots & 0 \\ 0 & 0 & B_3(G_3) & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & B_k(G_k) \end{bmatrix}.$$

We know rank $B_i = m_i - n_i + 1$, for $1 \le i \le k$.

Therefore, rank $B =$ rank $B_1 + \ldots +$ rank B_k

$$= (m_1 - n_1 + 1) + \ldots + (m_k - n_k + 1)$$

$$= (m_1 + \ldots + m_k) - (n_1 + \ldots + n_k) + k = m - n + k.$$ ❑

Definition: Let A be a matrix of order $k \times m$, with $k < m$. The *major determinant* of A is the determinant of the largest square submatrix of A, formed by taking any k columns of A. That is, the determinant of any $k \times k$ square submatrix is called the major determinant of A.

Let A and B be matrices of orders $k \times m$ and $m \times k$ respectively $(k < m)$. If columns $i_1, i_2, \ldots, i_k$ of B are chosen for a particular major of B, then the corresponding major in A consists of the rows $i_1, i_2, \ldots, i_k$ in A.

If A is a square matrix of order n, then $AX = 0$ has a non trivial solution $X \neq 0$ if and only if A is singular, that is $|A| = 0$. The set of all vectors X that satisfy $AX = 0$ forms a vector space called the null space of matrix A. The rank of the null space is called the nullity of A. Further more,

$$\text{rank } A + nullity A = n.$$

These definitions and the above equation also hold when A is a matrix of order $k \times n,\ k < n$.

We now give Binet-Cauchy and Sylvester theorems which will be used in the further discussions.

Theorem 10.12 (Binet−Cauchy) If A and B are two matrices of the order $k \times m$ and $m \times k$ respectively $(k < m)$, then $|AB| =$ sum of the products of corresponding major determinants of A and B.

Proof We multiply two $(m+k) \times (m+k)$ partitioned matrices to get

$$\begin{bmatrix} I_k & A \\ O & I_m \end{bmatrix} \begin{bmatrix} A & O \\ -I_m & B \end{bmatrix} = \begin{bmatrix} O & AB \\ -I_m & B \end{bmatrix},$$

where I_m and I_k are identity matrices of order m and k respectively.

Therefore, $\det \begin{bmatrix} A & O \\ -I_m & B \end{bmatrix} = \det \begin{bmatrix} O & AB \\ -I_m & B \end{bmatrix}$.

Thus, $\det(AB) = \det \begin{bmatrix} A & O \\ -I_m & B \end{bmatrix}$. (10.12.1)

Now apply Cauchy's expansion method to the right side of (10.12.1) and observe that the only non-zero minors of any order in $-I_m$ are its principal minors of that order. Therefore, we see that the Cauchy expansion consists of these minors of order $m-k$ multiplied by their cofactors of order k in A and B together. ❑

Theorem 10.13 (Sylvester) If A and B are matrices of order $k \times m$ and $n \times p$ respectively, then nullity $AB \leq$ nullity $A +$ nullity B.

Proof Since every vector X satisfying $BX = 0$ also satisfies $ABX = 0$, therefore we have

$$\text{nullity } AB \geq \text{nullity } B \geq 0. \qquad (10.13.1)$$

Let nullity $B = s$. So there exists a set of s linearly independent vectors $\{x_1, x_2, \ldots, x_s\}$ forming a basis of the null space of B. Therefore,

$$BX_i = 0, \text{ for } i = 1, 2, \ldots, s. \tag{10.13.2}$$

Now let nullity $AB = s+t$. Thus there exists a set of t linearly independent vectors $[X_{s+1}, X_{s+2}, \ldots, X_{s+t}]$ such that the set $\{X_1, X_2, \ldots, X_s, X_{s+1}, \ldots, X_{s+t}\}$ forms a basis for the null space of AB. Therefore,

$$ABX_i = 0, \text{ for } i = 1, 2, \ldots, s, s+1, \ldots, s+t. \tag{10.13.3}$$

This implies that out of the $s+t$ vectors X_i forming a basis of the null space of AB, the first s vectors are made zero by B and the remaining non-zero BX_i'$s, i = s+1, \ldots, s+t$ are made zero by A.

Clearly, the vectors $BX_{s+1}, \ldots, BX_{s+t}$ are linearly independent. For if

$$b_1BX_{s+1} + b_2BX_{s+2} + \ldots + b_tBX_{s+t} = 0,$$

i.e., if $B(b_1X_{s+1} + b_2X_{s+2} + \ldots + b_tX_{s+t}) = 0$,

then the vector $b_1X_{s+1} + b_2X_{s+2} + \ldots + b_tX_{s+t}$ is the null space of B, which is possible only if $b_1 = b_2 = \ldots = b_t = 0$.

Thus we have seen that there are at least t linearly independent vectors which are made zero by A. So, nullity $A \geq t$.

Since $t = (s+t) - s$, therefore $t =$ nullity $AB -$ nullity B

Therefore, nullity $AB-$ nullity $B \leq$ nullity A, and so

$$\text{nullity } AB \leq \text{nullity } A + \text{nullity } B. \tag{10.13.4}$$ ❑

Corollary 10.2 We know, rank A + nullity $A = n$, and using this in (10.13.4), we get

$$n- \text{rank } AB \leq n- \text{rank } A + n- \text{rank } B.$$

Therefore, rank $AB \geq$ rank $A+$ rank $B - n$.

If in above, $AB = 0$, then rank A + rank $B \leq n$.

10.4 Cut-Set Matrix

Let G be a graph with m edges and q cutsets. The cut-set matrix $C = [c_{ij}]_{q\times m}$ of G is a (0, 1)-matrix with

$$c_{ij} = \begin{cases} 1, & if\ ith\ cutset\ contains\ jth\ edge, \\ 0, & otherwise. \end{cases}$$

Example Consider the graphs shown in Figure 10. 7.

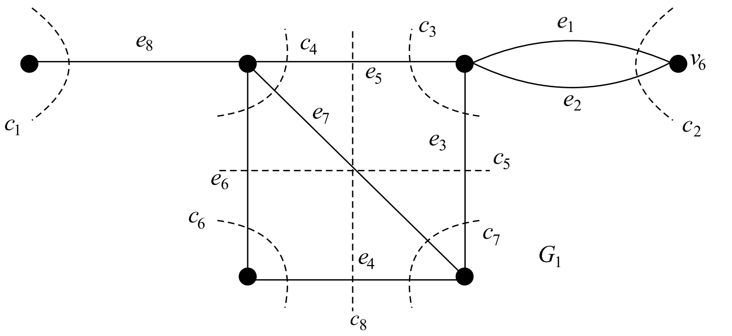

Fig. 10.7(a)

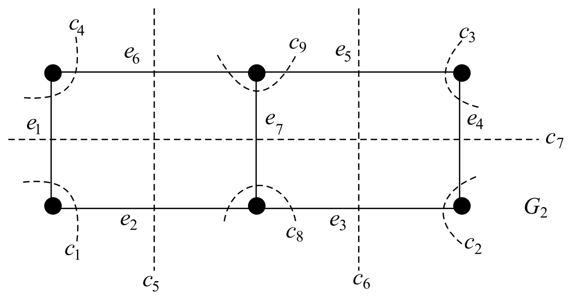

Fig. 10.7(b)

In the graph G_1, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

The cut-sets are $c_1 = \{e_8\}$, $c_2 = \{e_1, e_2\}$, $c_3 = \{e_3, e_5\}$, $c_4 = \{e_5, e_6, e_7\}$, $c_5 = \{e_3, e_6, e_7\}$, $c_6 = \{e_4, e_6\}$, $c_7 = \{e_3, e_4, e_7\}$ and $c_8 = \{e_4, e_5, e_7\}$.

The cut-sets for the graph G_2 are $c_1 = \{e_1, e_2\}$, $c_2 = \{e_3, e_4\}$, $c_3 = \{e_4, e_5\}$, $c_4 = \{e_1, e_6\}$, $c_5 = \{e_2, e_6\}$, $c_6 = \{e_3, e_5\}$, $c_7 = \{e_1, e_4, c_7\}$, $c_8 = \{e_2, e_3, e_7\}$ and $c_9 = \{e_5, e_6, e_7\}$.

Thus the cut-set matrices are given by

$$C(G_1) = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}, \text{ and}$$

$$C(G_2) = \begin{array}{cc} & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

We have the following observations about the cut-set matrix $C(G)$ of a graph G.

1. The permutation of rows or columns in a cut-set matrix corresponds simply to renaming of the cut-sets and edges respectively.
2. Each row in $C(G)$ is a cut-set vector.
3. A column with all zeros corresponds to an edge forming a self-loop.
4. Parallel edges form identical columns in the cut-set matrix.
5. In a non-separable graph, since every set of edges incident on a vertex is a cut-set, therefore every row of incidence matrix $A(G)$ is included as a row in the cut-set matrix $C(G)$. That is, for a non-separable graph G, $C(G)$ contains $A(G)$. For a separable graph, the incidence matrix of each block is contained in the cut-set matrix. For example, in the graph G_1 of Figure 10.7, the incidence matrix of the block $\{e_3, e_4, e_5, e_6, e_7\}$ is the 4×5 submatrix of C, left after deleting rows c_1, c_2, c_5, c_8 and columns e_1, e_2, e_8.
6. It follows from observation 5, that rank $C(G) \geq$ rank $A(G)$. Therefore, for a connected graph with n vertices, rank $C(G) \geq n-1$.

The following result for connected graphs shows that cutset matrix, incidence matrix and the corresponding graph matrix have the same rank.

Theorem 10.14 If G is a connected graph, then the rank of a cut-set matrix $C(G)$ is equal to the rank of incidence matrix $A(G)$, which equals the rank of graph G.

Proof Let $A(G)$, $B(G)$ and $C(G)$ be the incidence, cycle and cut-set matrix of the connected graph G. Then we have

$$\text{rank } C(G) \geq n-1. \tag{10.14.1}$$

Since the number of edges common to a cut-set and a cycle is always even, every row in C is orthogonal to every row in B, provided the edges in both B and C are arranged in the same order.

Thus, $BC^T = CB^T \equiv 0 \pmod 2$. (10.14.2)

Now, applying Sylvester's theorem to equation (10.14.2), we have

rank $B+$ rank $C \leq m$.

For a connected graph, we have rank $B = m - n + 1$.

Therefore, rank $C \leq m-$ rank $B = m - (m - n + 1) = n - 1$.

So, rank $C \leq n - 1$. (10.14.3)

It follows from (10.14.1) and (10.14.3) that rank $C = n - 1$. ❑

10.5 Fundamental Cut-Set Matrix

Let G be a connected graph with n vertices and m edges. The fundamental cut-set matrix C_f of G is an $(n-1) \times m$ submatrix of C such that the rows correspond to the set of fundamental cut-sets with respect to some spanning tree. Clearly, a fundamental cut-set matrix C_f can be partitioned into two submatrices, one of which is an identity matrix I_{n-1} of order $n-1$. We have

$$C_f = [C_c : I_{n-1}],$$

where the last $n-1$ columns forming the identity matrix correspond to the $n-1$ branches of the spanning tree and the first $m-n+1$ columns forming C_c correspond to the chords.

Example Consider the connected graphs G_1 and G_2 given in Figure 10.8. The spanning tree is shown with bold lines. The fundamental cut-sets of G_1 are c_1, c_2, c_3, c_6 and c_7 while the fundamental cut-sets of G_2 are c_1, c_2, c_3, c_4 and c_7.

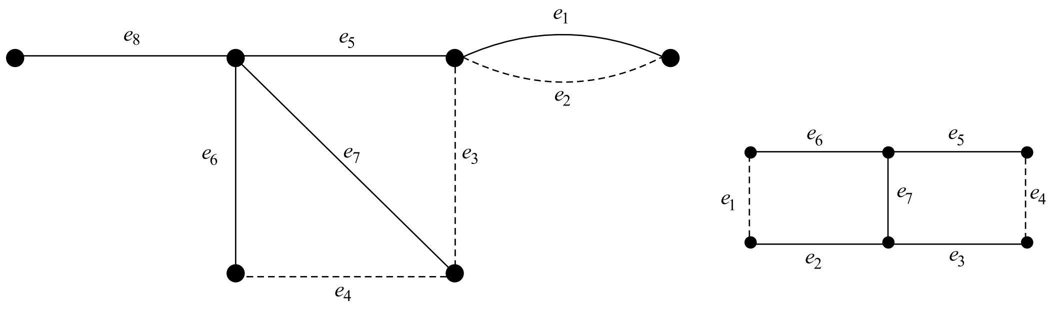

Fig. 10.8

The fundamental cut-set matrix of G_1 and G_2, respectively are given by

$$C_f = \begin{array}{c} \begin{array}{cccccccc} e_2 & e_3 & e_4 & e_1 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \left[\begin{array}{ccc:ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \text{ and}$$

$$C_f = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_4 & e_2 & e_3 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{cc:ccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

10.6 Relations between A_f, B_f and C_f

Let G be a connected graph and A_f, B_f and C_f be respectively the reduced incidence matrix, the fundamental cycle matrix, and the fundamental cut-set matrix of G.

We have shown that

$$B_f = \left[I_\mu \vdots B_t\right] \tag{10.6.i}$$

and $$C_f = \left[C_c \vdots I_{n-1}\right], \tag{10.6.ii}$$

where B_t denotes the submatrix corresponding to the branches of a spanning tree and C_c denotes the submatrix corresponding to the chords.

Let the spanning tree T in Equations (10.6.i) and (10.6.ii) be the same and let the order of the edges in both equations be same. Also, in the reduced incidence matrix A_f of size $(n-1) \times m$, let the edges (i.e., the columns) be arranged in the same order as in B_f and C_f.

Partition A_f into two submatrices given by

$$A_f = \left[A_c \vdots A_t\right], \tag{10.6.iii}$$

where A_t consists of $n-1$ columns corresponding to the branches of the spanning tree T and A_c is the spanning submatrix corresponding to the $m-n+1$ chords.

Since the columns in A_f and B_f are arranged in the same order, the equation $AB^T = BA^T = 0(\text{mod } 2)$ gives

$$A_f B_f^T \equiv 0(\text{mod } 2),$$

or $$\begin{bmatrix} A_c \vdots A_t \end{bmatrix} \begin{bmatrix} I_\mu \\ \vdots \\ B_t^T \end{bmatrix} \equiv 0(\text{mod } 2),$$

or $$A_c + A_t B_f^T \equiv 0(\text{mod } 2). \tag{10.6.iv}$$

Since A_t is non singular, A_t^{-1} exists. Now, premultiplying both sides of equation (10.6.iv) by A_t^{-1}, we have

$$A_t^{-1} A_c + A_t^{-1} A_t B_t^T \equiv 0(\text{mod } 2),$$

or $$A_t^{-1} A_c + B_t^T \equiv 0(\text{mod } 2).$$

Therefore, $A_t^{-1} A_c = -B_t^T$.

Since in mod 2 arithmetic $-1 = 1$,

$$B_t^T = A_t^{-1} A_c. \tag{10.6.v}$$

Now as the columns in B_f and C_f are arranged in the same order, therefore (in mod 2 arithmetic) $C_f . B_f^T \equiv 0(\text{mod } 2)$ in mod 2 arithmetic gives $C_f . B_f^T = 0$.

Therefore, $\begin{bmatrix} C_c \vdots I_{n-1} \end{bmatrix} \begin{bmatrix} I_\mu \\ \vdots \\ B_t^T \end{bmatrix} = 0$, so that $C_c + B_t^T = 0$, that is, $C_c = -B_t^T$.

Thus, $C_c = B_t^T$ (as $-1 = 1$ in mod 2 arithmetic).

Hence, $C_c = A_t^{-1} A_c$ from (10.6.v).

Remarks We make the following observations from the above relations.

1. If A or A_f is given, we can construct B_f and C_f starting from an arbitrary spanning tree and its submatrix A_t in A_f.
2. If either B_f or C_f is given, we can construct the other. Therefore, since B_f determines a graph within 2-isomorphism, so does C_f.
3. If either B_f and C_f is given, then A_f in general cannot be determined completely.

Example Consider the graph G of Figure 10.9.

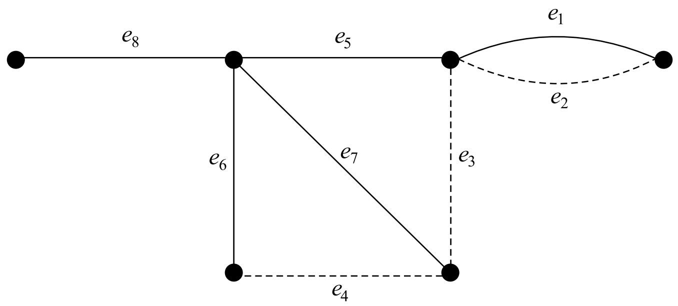

Fig. 10.9

Let $\{e_1, e_5, e_6, e_7, e_8\}$ be the spanning tree.

$$\text{We have, } A = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

Dropping the sixth row in A, we get

$$A_f = \begin{array}{c} \begin{matrix} e_2 & e_3 & e_4 & & e_1 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & : & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & : & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & : & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & : & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & : & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = [A_c : A_t].$$

$$B_f = \begin{array}{c} \begin{matrix} e_2 & e_3 & e_4 & & e_1 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = [I_3 : B_t] \text{ and}$$

$$C_f = \begin{array}{c} \begin{array}{cccccccc} e_2 & e_3 & e_4 & e_1 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = [C_c : I_5].$$

Clearly, $B_t^T = C_c$.

We verify $A_t^{-1}A_c = B_t^T$.

Now,

$$A_t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Therefore, $A_t^{-1}Ac = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, $A_t^{-1}A_c = B_t^T$.

10.7 Path Matrix

Let G be a graph with m edges, and u and v be any two vertices in G. The path matrix for vertices u and v denoted by $P(u, v) = [p_{ij}]_{q \times m}$, where q is the number of different paths between u and v, is defined as

$$p_{ij} = \begin{cases} 1, & if\ jth\ edge\ lies\ in\ the\ ith\ path, \\ 0, & otherwise. \end{cases}$$

Clearly, a path matrix is defined for a particular pair of vertices, the rows in $P(u, v)$ correspond to different paths between u and v, and the columns correspond to different edges in G. For example, consider the graph in Figure 10.10.

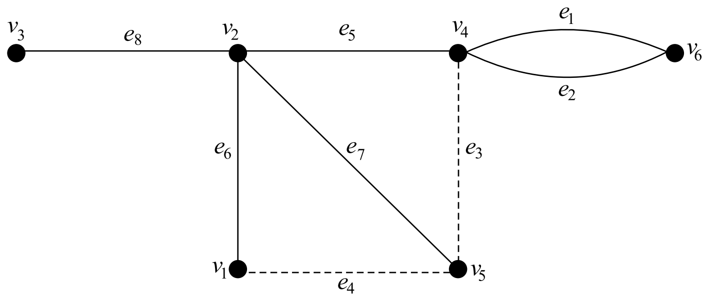

Fig. 10.10

The different paths between the vertices v_3 and v_4 are

$$p_1 = \{e_8, e_5\}, \; p_2 = \{e_8, e_7, e_3\} \text{ and } p_3 = \{e_8, e_6, e_4, e_3\}.$$

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

We have the following observations about the path matrix.

1. A column of all zeros corresponds to an edge that does not lie in any path between u and v.
2. A column of all ones corresponds to an edge that lies in every path between u and v.
3. There is no row with all zeros.
4. The ring sum of any two rows in $P(u, v)$ corresponds to a cycle or an edge-disjoint union of cycles.

The next result gives a relation between incidence and path matrix of a graph.

Theorem 10.15 If the columns of the incidence matrix A and the path matrix $P(u, v)$ of a connected graph are arranged in the same order, then under the product (mod 2).

$$AP^T(u, v) = M,$$

where M is a matrix having ones in two rows u and v, and the zeros in the remaining $n-2$ rows.

Proof Let G be a connected graph and let $v_k = u$ and $v_t = v$ be any two vertices of G. Let A be the incidence matrix and $P(u, v)$ be the path matrix of (u, v) in G.

Now for any vertex v_i in G and for any $u-v$ path p_j in G, either $v_i \in p_j$ or $v_i \notin p_j$.

If $v_i \notin p_j$, then there is no edge of p_j which is incident on v_i.

If $v_i \in p_j$, then either v_i is an intermediate vertex of p_j , or $v_i = v_k$ or v_t. In case v_i is an intermediate vertex of p_j, then there are exactly two edges of p_j which are incident on v_i and in case $v_i = v_k$ or v_t, there is exactly one edge of p_j which is incident on v_i.

Now consider the ith row of A and the jth row of P (which is the jth column of $P^T(u,\, v)$).

As the edges are arranged in the same order, the rth entries in these two rows are both non zero if and only if the edge e_r is incident on the ith vertex v_i and is also on the jth path p_j. Let $AP^T(u,\, v) = M = [m_{ij}]$.

We have, $\left[AP^T\right]_{ij} = \sum\limits_{r=1}^{m} [A]_{ir}[P^T]_{rj}$.

Therefore, $m_{ij} = \sum\limits_{r=1}^{m} a_{ir}p_{jr}$.

For each edge e_r of G, we have one of the following cases.

i. e_r is incident on v_i and $e_r \notin p_j$. Here $a_{ir} = 1, b_{jr} = 0$.

ii. e_r is not incident on v_i and $e_r \in p_j$. Here $a_{ir} = 0, b_{jr} = 1$.

iii. e_r is not incident on v_i and $e_r \notin p_j$. Here $a_{ir} = 0, b_{jr} = 0$.

All these cases imply that the ith vertex v_i is not in jth path p_j and we have $M_{ij} = 0 \equiv 0 (\text{mod } 2)$. (Fig. 10.11(a)).

iv. e_r is incident on v_i and $e_r \in p_j$ (Fig. 10.11(b)).

If v_i is an intermediate vertex of p_j, then there are exactly two edges say e_r and e_t incident on v_i so that $a_{ir} = 1,\ a_{it} = 1,\ p_{jr} = 1,\ p_{jt} = 1$.

Therefore, $m_{ij} = 1+1 = 0 (\text{mod } 2)$.

If $v_i = v_k$ or v_t then the edge e_r is incident on either v_k or v_t. So, $a_{kr} = 1, p_{jr} = 1$, or $a_{tr} = 1, p_{jr} = 1$.

Thus, $m_{kj} = \Sigma a_{ir}p_{jr} = 1.1 \equiv 1 (\text{mod } 2)$, and

$$m_{tj} = \Sigma a_{ir}p_{jr} = 1.1 \equiv 1 (\text{mod } 2).$$

Hence $M = [m_{ij}]$ is a matrix, such that under modulo 2,

$$m_{ij} = \begin{cases} 1\,, & for\ i = k,\, t\,, \\ 0\,, & otherwise\,. \end{cases}$$ ❑

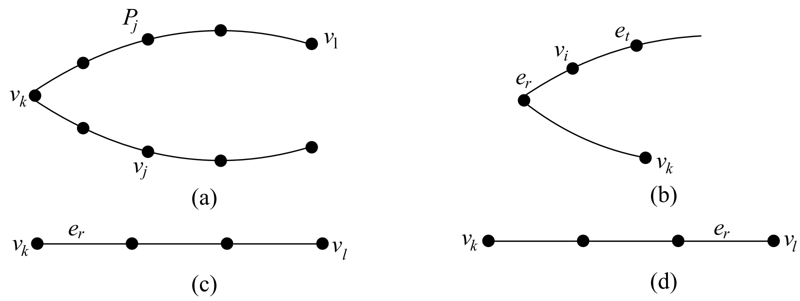

Fig. 10.11

Example In the graph of Figure 10.10, we have

$$AP^T(v_3, v_4) = \begin{bmatrix} 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&1\\ 1&1&1&0&1&0&0&0\\ 0&0&1&1&0&0&1&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&1&1\\ 0&0&1\\ 1&0&0\\ 0&0&1\\ 0&1&0\\ 1&1&1 \end{bmatrix} = \begin{array}{c} v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6 \end{array} \begin{bmatrix} 0&0&0\\ 0&0&0\\ 1&1&1\\ 1&1&1\\ 0&0&0\\ 0&0&0 \end{bmatrix} \text{(mod2)}.$$

10.8 Adjacency Matrix

Let $V = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ and without parallel edges. The adjacency matrix of G is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1, & if\ v_i v_j \in E, \\ 0, & otherwise. \end{cases}$$

Example Consider the graph G given in Figure 10.12.

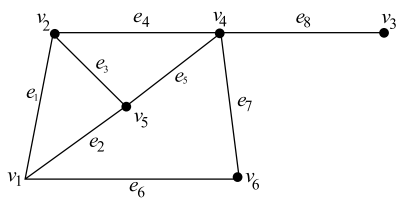

Fig. 10.12

The adjacency matrix of G is given by

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

We have the following observations about the adjacency matrix X of a graph G.

1. The entries along the principal diagonal of X are all zeros if and only if the graph has no self-loops. However, a self-loop at the ith vertex corresponds to $x_{ii} = 1$.

2. If the graph has no self-loops, the degree of a vertex equals the number of ones in the corresponding row or column of X.

3. Permutation of rows and the corresponding columns imply reordering the vertices. We note that the rows and columns are arranged in the same order. Therefore, when two rows are interchanged in X, the corresponding columns are also interchanged. Thus two graphs G_1 and G_2 without parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related by

$$X(G_2) = R^{-1}X(G_1)R,$$

where R is a permutation matrix.

4. A graph G is disconnected having components G_1 and G_2 if and only if the adjacency matrix $X(G)$ is partitioned as

$$X(G) = \begin{bmatrix} X(G_1) & : & O \\ .. & : & .. \\ O & : & X(G_2) \end{bmatrix},$$

where $X(G_1)$ and $X(G_2)$ are respectively the adjacency matrices of the components G_1 and G_2. Obviously, the above partitioning implies that there are no edges between vertices in G_1 and vertices in G_2.

5. If any square, symmetric and binary matrix Q of order n is given, then there exists a graph G with n vertices and without parallel edges whose adjacency matrix is Q.

Definition: An *edge sequence* is a sequence of edges in which each edge, except the first and the last, has one vertex in common with the edge preceding it and one vertex in common with the edge following it. A walk and a path are the examples of an edge sequence. An edge can appear more than once in an edge sequence. In the graph of Figure 10.13, $v_1e_1v_2e_2v_3e_3v_4e_4v_2e_2v_3e_5v_5$, or $e_1e_2e_3e_4e_2e_5$ is an edge sequence.

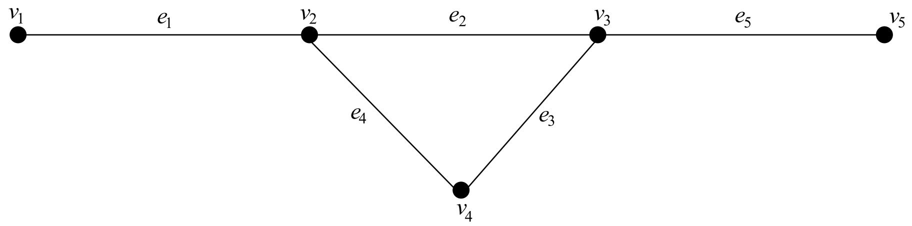

Fig. 10.13

We now have the following result.

Theorem 10.16 If $X = [x_{ij}]$ is the adjacency matrix of a simple graph G, then $[X^k]_{ij}$ is the number of different edge sequences of length k between vertices v_i and v_j.

Proof We prove the result by using induction on k. The result is trivial for $k = 0$ and 1. Since $X^2 = X.X$, X^2 is a symmetric matrix, as product of symmetric matrices is also symmetric.

For $k = 2, i \neq j$, we have

$[X^2]_{ij}$ = number of ones in the product of ith row and jth column (or jth row) of X

= number of positions in which both ith and jth rows of X have ones

= number of vertices that are adjacent to both ith and jth vertices

= number of different paths of length two between ith and jth vertices

Also, $[X^2]_{ii}$ = number of ones in the ith row (or column) of X

= degree of the corresponding vertex.

This shows that $[X^2]_{ij}$ is the number of different paths and therefore different edge sequences of length 2 between the vertices v_i and v_j. Thus the result is true for $k = 2$.

Assume the result to be true for k, so that

$[X^k]_{ij}$= number of different edge sequences of length k between v_i and v_j.

We have, $[X^{k+1}]_{ij} = [X^kX]_{ij} = \sum\limits_{r=1}^{n} [X^k]_{ir}[X]_{r_j} = \sum\limits_{r=1}^{n} [X^k]_{i_r}x_{r_j}$.

Now, every $v_i - v_j$ edge sequence of length $k+1$ consists of a $v_i - v_r$ edge sequence of length k, followed by an edge $v_r v_j$. Since there are $[X^k]_{ir}$ such edge sequences of length k and x_{rj} such edges for each vertex v_r, the total number of all $v_i - v_j$ edge sequences of length $k+1$ is $\sum_{r=1}^{n} [X^k]_{ir} x_{rj}$. This proves the result for $k+1$ also. ❑

We have the following observation about connectedness and adjacency matrix.

Theorem 10.17 Let G be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and let X be the adjacency matrix of G. Let $Y = [y_{ij}]$ be the matrix $Y = X + X^2 + \ldots + X^{n-1}$.

Then G is connected if and only if for all distinct i, j, $y_{ij} \neq 0$. That is, if and only if Y has no zero entries off the main diagonal.

Proof We have, $y_{ij} = [Y]_{ij} = [X]_{ij} + [X^2]_{ij} + \ldots + [X^{n-1}]_{ij}$.

Since $[X^k]_{ij}$ denotes the number of distinct edge-sequences of length k from v_i to v_j,

$y_{ij} =$ number of different $v_i - v_j$ edge sequence of length 1

$+$ number of different $v_i - v_j$ edge sequences of length $2 + \ldots$

$+$ number of different $v_i - v_j$ edge sequences of length $n-1$.

Therefore, $y_{ij} =$ number of different $v_i - v_j$ edge sequence of length less than n.

Now let G be connected. Then for every pair of distinct i, j there is a path from v_i to v_j. Since G has n vertices, this path passes through atmost n vertices and so has length less than n. Thus, $y_{ij} \neq 0$ for each i, j with $i \neq j$.

Conversely, for each distinct pair i, j we have $y_{ij} \neq 0$. Then from above, there is at least one edge sequence of length less than n from v_i to v_j. This implies that v_i is connected to v_j. Since the distinct pair i, j is chosen arbitrarily, G is connected. ❑

The next result is useful in determining the distances between different pairs of vertices.

Theorem 10.18 In a connected graph, the distance between two vertices v_i and v_j is k if and only if k is the smallest integer for which $[X^k]_{ij} \neq 0$.

Proof Let G be a connected graph and let $X = [x_{ij}]$ be the adjacency matrix of G. Let v_i and v_j be vertices in G such that

$d(v_i, v_j) = k$.

Then the length of the shortest path between v_i and v_j is k.

This implies that there are no paths of length $1, 2, \ldots, k-1$ and so no edge sequences of length $1, 2, \ldots, k-1$ between v_i and v_j.

Therefore, $[X]_{ij}=0,\ [X^2]_{ij}=0,\ \ldots,\ [X^{k-1}]_{ij}=0$.

Hence k is the smallest integer such that $[X^k]_{ij}\neq 0$.

Conversely, suppose that k is the smallest integer such that $[X^k]_{ij}\neq 0$.

Therefore, there are no edge sequences of length $1,\ 2,\ \ldots,\ k-1$ and in fact no paths of length $1,\ 2,\ \ldots,\ k-1$ between vertices v_i and v_j.

Thus the shortest path between v_i and v_j is of length k, so that $d(v_i,\ v_j)=k$. ❑

Definition: Let G be a graph and let d_i be the degree of the vertex v_i in G. The degree matrix $H=[h_{ij}]$ of G is defined by

$$h_{ij}=\begin{cases} 0, & for\ i\neq j\,, \\ d_i, & for\ i=j\,. \end{cases}$$

The following result gives a relation between the matrices F, X and H.

Theorem 10.19 Let F be the modified incidence matrix, X the adjacency matrix and H the degree matrix of a graph G. Then

$$FF^T=H-X.$$

Proof We have $(i,\ j)$th element of FF^T,

$$[FF^T]_{ij}=\sum_{r=1}^{m}[F]_{ir}[F^T]_{rj}=\sum_{r=1}^{m}[F]_{ir}[F]_{jr}.$$

Now, $[F]_{ir}$ and $[F]_{rj}$ are non-zero if and only if the edge $e_r=v_iv_j$. Then for $i\neq j$,

$$\sum_{r=1}^{m}[F]_{ir}[F]_{jr}=\begin{cases} -1, & if\ e_r=v_iv_j\ is\ an\ edge\,, \\ 0, & if\ e_r=v_iv_j\ is\ not\ an\ edge\,. \end{cases}$$

For $i=j$, $[F]_{ir}[F]_{jr}=1$ whenever $[F]_{ik}=\pm 1$, and this occurs d_i times corresponding to the number of edges incident on v_i. Thus,

$$\sum_{r=1}^{m}[F]_{ir}[F]_{jr}=d_i, \text{ for } i=j.$$

Therefore,

$$[FF^T]_{ij}=\begin{cases} -1\ or\ 0, & according\ to\ whether\ for\ i\neq j\,,\ v_iv_j\ is\ an\ edge\ or\ not\,, \\ d_i, & for\ i=j\,. \end{cases}$$

Also, $[H-X]_{ij} = [H]_{ij} - [X]_{ij}$

$$= \begin{cases} d_i - 0\,, & for\ i = j\,, \\ 0 - (1\ or\ 0)\,, & according\ as\ for\ i \neq j\,,\ v_iv_j\ \ is\ an\ edge,\ or\ v_iv_j\ \ is\ not\ an\ edge. \end{cases}$$

$$= \begin{cases} d_i\,, & for\ i = j\,, \\ -1\ or\ 0\,, & according\ as\ for\ i \neq j\,,\ v_iv_j\ \ is\ an\ edge,\ or\ v_iv_j\ \ is\ not\ an\ edge. \end{cases}$$

Hence $FF^T = H - X$. ❑

Corollary 10.3 The matrix $Q = FF^T$ is independent of the orientation used for the edges of G in getting F.

Theorem 10.20 Let X, F and H be the adjacency, modified incidence and degree matrices of the graph G, and $Q = FF^T = H - X$. Then the matrix of cofactors of Q denoted by adj Q is a multiple of the all ones $n \times n$ matrix J.

Proof If G is disconnected, then rank Q =rank $FF^T < n-1$ and so every cofactor of Q is zero. Therefore, adj $Q = O = O.J$, where $J = [J_{ij}]_{n \times n}$ with $J_{ij} = 1$ for all $i,\ j$.

Now let G be connected. Then rank $Q = n-1$ and therefore $|Q| = 0$ This implies that every column of adj Q belongs to the kernel (null space) of Q.

But nullity $Q = 1$ (as rank $Q+$ nullity $Q = n$). So, if u is the n-vector of ones, then

$$(H - X)u = 0.$$

and therefore u is in the null space of Q.

Thus every other vector in the null space of Q and in particular every column of adj Q is a multiple of u.

Since Q and so adj Q are symmetric, the multiplying factor for all columns of adj Q are same.

Hence, adj $Q = cJ$, where c is a constant. ❑

The next result called Matrix-tree theorem can be used in finding the complexity of a connected graph.

Theorem 10.21 (Matrix-tree theorem) If X, F and H are the adjacency, modified incidence and degree matrices of the connected graph G, and $Q = FF^T, J$ is the $n \times n$ matrix of ones and $\tau(G)$ is the complexity of G, then adj $Q = \tau(G).J$.

Proof Let X, F and H be the adjacency, modified incidence and degree matrix of a connected graph G. We have $Q = FF^T$ and adj $Q =$ matrix of the cofactors of Q. Also $\tau(G)J$ is a matrix whose every entry is $\tau(G)$ as J is a matrix whose every entry is unity.

Therefore, to prove adj $Q = \tau(G)J$, it is enough to prove that $\tau(G) =$ any one cofactor of Q.

Let F_0 be the matrix obtained by dropping the last row from F. Then clearly, $|F_o F_o^T|$ is a cofactor of Q.

Using Binet-Cauchy theorem of matrix theory, we have

$$\left|F_0 F_0^T\right| = \sum_{X \subseteq E} |F_x| \left|F_x^T\right|, \tag{10.21.1}$$

where F_x is the square submatrix of F_o whose $n-1$ columns correspond to $n-1$ edges in the subset X of E, the summation running over all possible such subsets.

We know $|F_x| \neq 0$ if and only if $< X >$ is a spanning tree of G and $|Fx| = \pm 1$.

But, $|F_x^T| = |F_x|$.

Therefore each $X \subseteq E$ such that $< X >$ is a spanning tree of G contributes one to the sum on the right of (10.21.1) and all other contributions are zero.

Hence $|F_o F_o^T| = \tau(G)$, proving the theorem. ❑

Corollary 10.4 Prove $\tau(K_n) = n^{n-2}$.

Proof Here, $Q = H - X = (n-1)I - (J-I) = nI - J$. Therefore,

$$Q = \begin{bmatrix} n & 0 & 0 & .. & 0 \\ 0 & n & 0 & .. & 0 \\ 0 & 0 & n & .. & 0 \\ \vdots \\ 0 & 0 & 0 & .. & n \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 \\ \vdots \\ 1 & 1 & 1 & .. & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n-1 & -1 & -1 & .. & -1 \\ -1 & n-1 & -1 & .. & -1 \\ -1 & -1 & n-1 & .. & -1 \\ \vdots \\ -1 & -1 & -1 & .. & n-1 \end{bmatrix}.$$

The cofactor of q_{11} is the $(n-1) \times (n-1)$ determinant given by

$$\text{cofactor of } q_{11} = \begin{vmatrix} n-1 & -1 & .. & -1 \\ -1 & n-1 & .. & -1 \\ \vdots \\ -1 & -1 & .. & n-1 \end{vmatrix}.$$

Subtracting the first row from each of the others and then adding the last $n-2$ columns to the first, we get

$$\text{cofactor of } q_{11} = \begin{vmatrix} 1 & -1 & -1 & .. & -1 \\ 0 & n & 0 & .. & 0 \\ 0 & 0 & n & .. & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & .. & n \end{vmatrix}.$$

Expanding with the help of the first column, we have cofactor of $q_{11} = n^{n-2}$. Thus,

$$\tau(k_n) = n^{n-2}.$$ ❑

10.9 Exercises

1. Characterise A_f, B_f, C_f and X of the complete graph of n vertices.
2. Characterise simple, self-dual graphs in terms of their cycle and cut-set matrices.
3. Show that each diagonal entry in X^3 equals twice the number of triangles passing through the corresponding vertex.
4. Characterise the adjacency matrix of a bipartite graph.
5. Prove that a graph is bipartite if and only if for all odd k, every diagonal entry of A^k is zero.
6. Similar to the cycle or cut-set matrix, define a spanning tree matrix for a connected graph, and observe some of its properties.
7. If X is the adjacency matrix of a graph G and L is the adjacency matrix of its edge graph $L(G)$ and A and H are the incidence and degree matrices, show that $X = AA^T - H$ and $L = A^T A - 2I$.
8. Use the matrix tree theorem to calculate $\tau(K_4 - e)$.
9. Prove that $\tau(G) = \dfrac{1}{n^2}\det(J + Q)$.