Module 11 Directed Graphs

Contents

Graphs that we studied in chapters 1 to 10 are inadequate to model many real world problems. These include one-way message routings and Turing machine computations. In all such problems one requires the notion of direction from one node to another node.

11.1 Basic concepts

Definition. *A* ***directed graph*** *D is a triple (V, A, I_D) where V and A are disjoint sets and $I_D : A \to V \times V$ is a function.*

- *An element of V is called a* ***vertex****.*
- *An element of A is called an* ***arc****.*

Often we call a directed graph as a ***digraph***. As in the case of graphs we assume that V and A are finite sets and denote $|V|$ by n and $|A|$ by m. If more than one graph are under discussion, we denote V, A, n and m by $V(D), A(D), n(D)$ and $m(D)$, respectively.

An example of a digraph:

Let $V = \{1, 2, 3, 4, 5\}$, $A = \{a, b, c, d, e, f, g, h\}$ and $I_D : A \to V \times V$ be defined by $I_D(a) = (1, 2)$, $I_D(b) = (2, 3)$, $I_D(c) = (3, 2)$, $I_D(d) = (4, 3)$, $I_D(e) = (4, 1)$, $I_D(f) = (4, 1)$, $I_D(g) = (3, 5)$, $I_D(h) = (5, 5)$. Then (V, A, I_D) is a digraph with 5 vertices and 8 arcs. It is represented as follows:

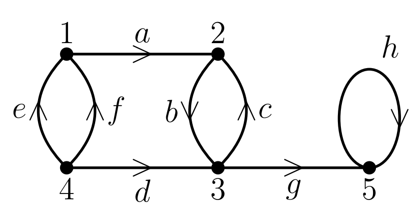


Figure 11.1: A digraph.

As in the case of graphs e and f are multiple arcs and h is a loop. However, b and c are not multiple arcs.

If $I_D(x) = (u, v)$, then x is denoted by $x(u, v)$ and say that

- ◦ u is the ***tail*** of x.
- ◦ v is the ***head*** of x.
- ◦ x is an arc ***from*** u ***to*** v.
- ◦ If $x(u, u)$ is a loop, then u is its head and also its tail.
- ◦ A directed graph with no multiple arcs and no loops is called a ***simple digraph***.

• Underlying graph of a digraph

The underlying graph $G(D)$ of a digraph $D(V, A, I_D)$ is obtained by ignoring the direction of its arcs. Formally, $V(G) = V$, $E(G) = A$ and $I_G : A \to V^{(2)}$ is defined by $I_G(a) = (u, v)$, if $I_D(a) = (u, v)$ or (v, u). The underlying graph of a digraph is shown below.

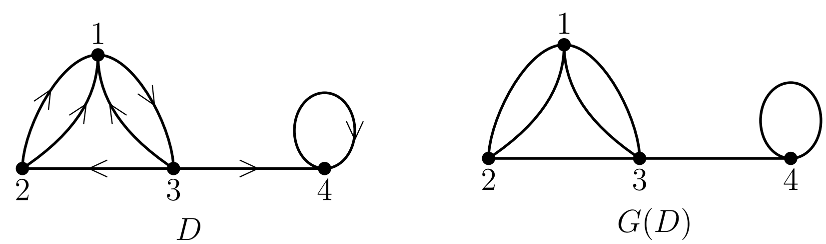


Figure 11.2: A digraph and its underlying graph.

Often we call a graph as an ***undirected graph*** if both a graph and a digraph are under discussion.

The various subdigraphs of a digraph D are defined analogous to subgraphs of a graph. We continue to use the notation $D - W$ (where $W \subseteq V(D)$) and $D - B$

(where $B \subseteq A(D)$) for the subdigraphs obtained from D by deleting a set of vertices and a set of arcs.

The concepts of degrees, walks and connectivity are defined taking into account the direction of arcs.

• Out-degrees and in-degrees

Definitions. *Let D be a digraph and v be a vertex.*

- *The **out-degree** of v is the number of arcs with v as their tail. It is denoted by $outdeg_D(v)$.*
- *The set*
$$N_{out}(v) = \{x \in V(D) : (v, x) \in A(D)\}$$
*is called the **set of out-neighbors** of v. Clearly, if D is **simple**, then $|N_{out}(v)| = outdeg(v)$.*
- *The **in-degree** of v is the number of arcs with v as their heads. It is denoted by $indeg_D(v)$.*
- *The set*
$$N_{in}(v) = \{x \in V(D) : (x, v) \in A(D)\}$$
*is called the **set of in-neighbors** of v. Clearly, if D is simple, then $|N_{in}(v)| = indeg(v)$.*

So each vertex v in a digraph is associated with an ordered pair $(outdeg(v), indeg(v))$ of integers; see Figure 11.3.

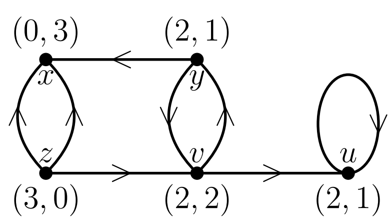


Figure 11.3: Out-degrees and in-degrees of vertices. $N_{out}(x) = \emptyset$, $N_{in}(x) = \{y, z\}$, $N_{out}(y) = \{x, v\}$, $N_{in}(y) = \{v\}$, $N_{out}(u) = \{u\}$, $N_{in}(u) = \{u, v\}$.

Theorem 11.1. *For every digraph D,*

$$\sum_{v \in V(D)} outdeg(v) = \sum_{v \in V(D)} indeg(v) = m.$$

Proof. Every arc is counted once in $\sum outdeg(v)$ and once in $\sum indeg(v)$. □

• Isomorphism

Definition. *Two digraphs $D_1(V_1, A_1, I_{D_1})$ and $D_2(V_2, A_2, I_{D_2})$ are said to be **isomorphic** if there exist bijections $f : V_1 \to V_2$ and $g : A_1 \to A_2$ such that x is an arc from u to v in D_1 if and only if $g(x)$ is an arc from $f(u)$ to $f(v)$ in D_2.*

*The pair of functions (f, g) is called an **Isomorphism**. If D_1 and D_2 are isomorphic, we write $D_1 \simeq D_2$.*

Figure 11.4 shows isomorphic and non-isomorphic digraphs.

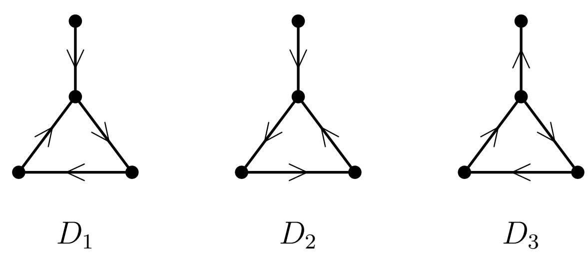


Figure 11.4: $D_1 \simeq D_2, D_1 \not\simeq D_3$.

Remark. If $D_1(V_1, A_1)$ and $D_2(V_2, A_2)$ are simple digraphs, then $D_1 \simeq D_2$ if and only if there exists a bijection $f : V_1 \to V_2$ such that $(u, v) \in A_1$ if and only if $(f(u), f(v)) \in A_2$.

11.2 Directed walks, paths and cycles

Definitions. *Let D be a digraph and let $v_0, v_t \in V(D)$.*

- *An alternating sequence*

$$W(v_0, v_t) := (v_0, a_1, v_1, a_2, v_2, \ldots, v_{t-1}, a_t, v_t)$$

*of vertices and arcs where a_i $(1 \le i \le t)$ is an arc from v_{i-1} to v_i is called a (v_0, v_t)-**directed walk** or a **directed walk from** v_0 **to** v_t. Here, the vertices or arcs need not be distinct.*

- *v_0 is called the **origin** and v_t is called the **terminus** of $W(v_0, v_t)$. Its length is defined to be t, the number of arcs, where an arc is counted as many times as it occurs.*

- *A (v_0, v_t)-walk is called a (v_0, v_t)-**trail** if all its arcs are distinct.*

- *A (v_0, v_t)-walk is called a (v_0, v_t)-**path** if all its vertices (and) hence arcs are distinct. A path is denoted by the sequence of vertices alone if no confusions is anticipated.*

- *A $W(v_0, v_t)$ is called a **closed directed walk** if $v_0 = v_t$.*

- *A closed directed walk $W(v_0, v_t)$ is called a **directed cycle** if all its vertices are distinct except that $v_0 = v_t$.*

Remark. In all these definitions, the adjective "directed" can be dropped if it is clear from the context that we are concerned with directed graphs.

We illustrate these concept by taking a digraph.

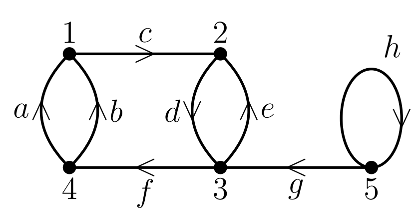


Figure 11.5: A digraph D.

(i) $W_1(5, 4) = (5, g, 3, e, 2, d, 3, e, 2, d, 3, f, 4)$
is a directed walk of length 6. It is neither a trail nor a path.

(ii) $W_2(5,4) = (5, g, 3, e, 2, d, 3, f, 4)$
is a directed trail of length 4. It is not a path.

(iii) $W_2(5,4) = (5, g, 3, f, 4)$
is a directed path of length 2.

(iv) $W_3(1, c, 2, d, 3, f, 4, b, 1)$
is a directed cycle of length 4.

Remarks.

- If there exists a (v_0, v_t)- directed walk, it is not necessary that, there exists a (v_t, v_0)-directed walk. In the above example, we have a (5,4)-directed walk but no (4,5)-directed walk.
- If there exists a (v_0, v_t)-directed walk, there exists a (v_0, v_t)-directed path.

• Connectivity in digraphs

Definitions.

- *A digraph is said to be* ***weakly connected*** *if its underlying graph is connected; otherwise, it is said to be disconnected.*
- *A digraph is said to be* ***unilaterally connected*** *if given any two vertices u and v, there exists a directed path from u to v or a directed path from v to u.*
- *A digraph is said to be* ***strongly connected*** *or* ***strong*** *if given any two vertices u and v, there exists a directed path from u to v and a directed path from v to u.*

Clearly, D is strongly connected $\Rightarrow$ D is unilaterally connected $\Rightarrow$ D is weakly connected. The converse implications do not hold; see Figure 11.6.

Theorem 11.2. *A digraph D is strong if and only if it contains a closed directed walk which contains all its vertices.*

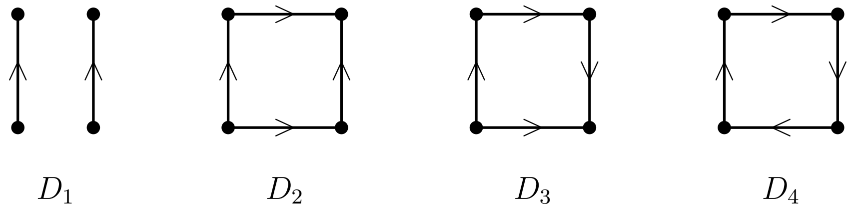


Figure 11.6: D_1 is disconnected; D_2 is weakly connected but it is not unilaterally connected; D_3 is unilaterally connected but it is not strongly connected; D_4 is strongly connected.

Proof. (1) D is strong $\Rightarrow$ D contains a closed directed walk which contains all its vertices.

Since D is strong, if $u, v \in V(D)$, then there exist directed paths $P_1(u,v)$ and $P_2(v,u)$. Therefore, D contains a closed directed walk $(P_1(u,v), P_2(v,u))$. Among all the closed directed walks, let $W(x,x)$ be a closed directed walk containing maximum number of vertices.

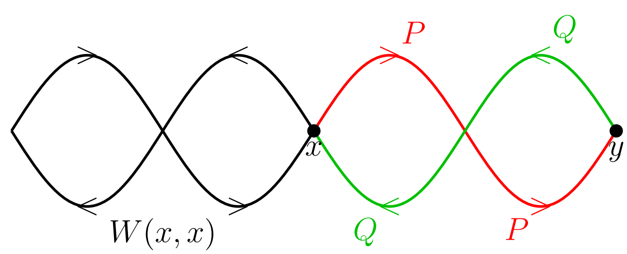


Figure 11.7: Extension of $W(x,x)$.

Our aim is to show that W contains all the vertices of D. On the contrary, suppose that there exists a vertex $y \in V(D) - V(W)$. Since D is strong, there exist directed paths $P(x,y)$ and $Q(x,y)$; see Figure 11.7. But then $(W(x,x), P(x,y), Q(y,x))$ is a closed directed walk containing more number of vertices than W, a contradiction to the maximality of W. Therefore, $W(x,x)$ contains all the vertices of D.

(2) D contains a closed directed walk which contains all the vertices of $D \Rightarrow D$ is strong.

Let $W(x, x)$ be a closed directed walk containing all the vertices of D. Let $u, v \in V(D) = V(W)$. Since W is closed, W contains directed subwalks $W_1(u, v)$ and $W_2(v, u)$. Hence D is strong. □

Theorem 11.3. *A digraph D is unilaterally connected if and only if it contains a directed walk (not necessarily closed) containing all the vertices of D.*

Proof. (1) D is unilateral $\Rightarrow D$ contains a directed walk containing all the vertices of D.

Among all the directed walks in D, let W be a directed walk containing maximum number of vertices of D. Let $W = W(u_0, u_t) = (u_0, e_1, u_1, e_2, u_2, \ldots, u_{t-1}, e_t, u_t)$. We assert that W contains all the vertices of D. On the contrary, suppose that there exists a vertex $x \in V(D) - V(W)$; see Figure 11.8.

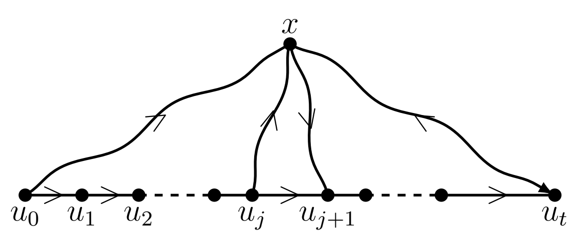


Figure 11.8: Extension of $W(u_0, u_t)$.

Claim 1: There exists a (u_0, x)-walk.

Else, there exists a (x, u_0)-walk, say $W'(x, u_0)$, since D is unilaterally connected. But then $(W'(x, u_0), W(u_0, u_t))$ is a walk in D containing more number of vertices than W, a contradiction to the maximality of W.

Claim 2: If there exists a (u_j, x)-walk for some j, $1 \leq j \leq t-1$, then there does not exist a (x, u_{j+1})-walk in D.

On the contrary, suppose there exist walks $W_j(u_j, x)$ and $W_{j+1}(x, u_{j+1})$. But then

$$(W(u_0, u_j), W_j(u_j, x), W_{j+1}(x, u_{j+1}), W(u_{j+1}, u_t))$$

is a (u_0, u_t)-walk containing more number of vertices than W, a contradiction to the maximality of W.

Claims 1 and 2 imply that there exist a (u_t, x)-walk, say $W'(u_t, x)$. But then $(W(u_0, u_t), W'(u_t, x))$ is a (u_0, x)-walk containing more number of vertices than W, a contradiction as before.

Therefore, $W(u_0, u_t)$ contains all the vertices of D.

(2) D contains a directed walk W containing all the vertices of $D \Rightarrow D$ is unilateral.

Let $u, v \in V(D) = V(W)$. Then W contains a directed (u, v)-subwalk if u precedes v in W or a (v, u)-subwalk if v precedes u in W. Hence D is unilateral. □

Theorem 11.4. *Let D be a simple digraph satisfying any one of the following two conditions for some integer p:*

(1) outdeg$(v) \geq p \geq 1$, for every vertex v.

(2) indeg$(v) \geq p \geq 1$, for every vertex v.

Then D contains a directed cycle of length $\geq p + 1$.

Proof. It is similar to the proof of Theorem 2.2. (Hint: choose a directed path of maximum length, say $P(x, y)$. If (1) holds, then look at $N_{out}(y)$, and if (2) holds, then look at $N_{in}(x)$.) □

Corollary. *If D is a digraph satisfying any of the conditions stated in the above theorem, then D contains a directed cycle.*

11.3 Orientation of a graph

Definition. *If G is a graph, then an* ***orientation*** *of G is a digraph $D(G)$ obtained by orienting each edge (x, y) of G from x to y or y to x but not in both directions.*

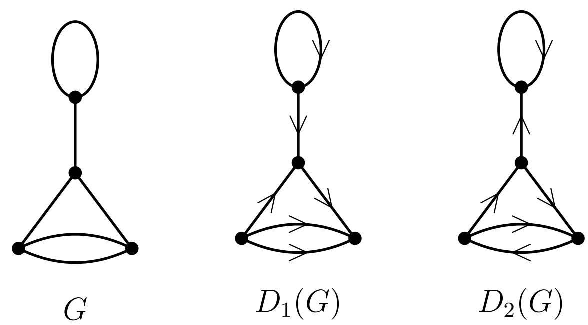


Figure 11.9: A graph G and two of its orientations.

Remark. Given a loopless graph G, there are 2^m orientations of G (some of which may be isomorphic).

Definition. An orientation $D(G)$ of a graph G is called a ***strong orientation*** if $D(G)$ is a strong digraph.

A graph G may not admit a strong orientation. For example, P_3 does not admit a strong orientation. (Why?) But K_3 admits a strong orientation.

A motivation for the definition of strong orientation is the following real-world problem.

When is it possible to make the roads of a city one-way in such a way that ***every corner of the city is reachable from every other corner ?***

Consider the road map of Figure 11.10a. It is impossible to make all the roads one-way as required. The impossibility is because of the road connecting the school and the garden.

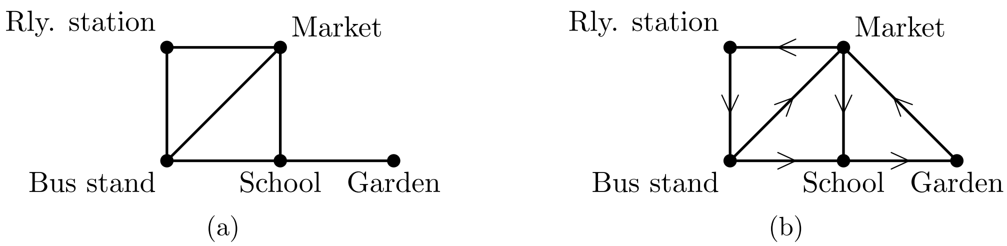


Figure 11.10: A road map.

However, if we have an additional road connecting the market and the garden, it is possible to make all the roads one-way as required (see Figure 11.10b.)

Definition. A graph G is said to be ***strongly orientable*** if it is possible to give an orientation to each edge of G so that the resulting digraph is strongly connected.

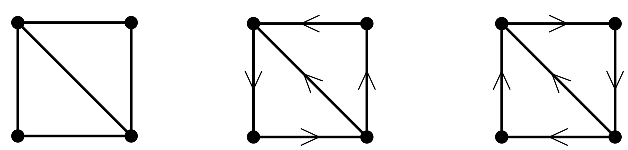


Figure 11.11: A graph G and its two strong orientations.

Thus in graph theoretic terminology, the one-way traffic problem is to characterize those graphs which are strongly orientable.

Theorem 11.5. *A connected graph G is strongly orientable if and only if G has no cut-edges.*

Proof. (1) G is strongly orientable $\Rightarrow$ G has no cut-edge.

Let D be a strong-orientation of G. Let (u, v) be an arbitrary edge of G. So, $(u, v) \in A(D)$ or $(v, u) \in A(D)$; say $(u, v) \in A(D)$. Since D is strongly connected, there exists a directed path $P(v,u)$. But then $(u,(u,v),P(v,u))$ is a cycle in G containing the edge (u,v). Hence by Theorem 2.15, (u,v) is not a cut-edge.

(2) G has no cut-edge $\Rightarrow G$ is strongly orientable.

Assume for the moment that a subgraph H with $V(G)-V(H)\neq\emptyset$ has been strongly oriented. Since $V(G)\neq V(H)$ and G is connected, there is an edge $e(u_0,u_1)\in E(G)-E(H)$, where $u_0\in V(H)$ and $u_1\in V(G)-V(H)$. Since e is not a cut-edge, there is a cycle $C(u_0,u_0)=(u_0,u_1,\ldots,u_p=u_0)$ in G containing e; see Figure 11.12.

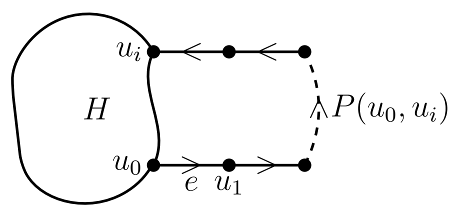


Figure 11.12: A step in the proof of Theorem 11.5.

Let u_i be the first vertex which succeeds u_1 and is in $V(H)$. (Note that u_i exists because $u_p=u_0$). Orient the path $(u_0,u_1,\ldots,u_i)$ from u to u_i so that it becomes a directed path $P(u,u_i)$. Clearly, the subdigraph H_1 which contains H and the directed path $P(u,u_i)$ is strongly connected. If we now arbitrarily orient the edges, whose end-vertices lie in $V(H)\cup\{u_0,u_1,\ldots,u_i\}$, we get a strong orientation of the subgraph induced on $V(H)\cup\{u_0,u_1,\ldots,u_i\}$. Moreover this new subdigraph contains at least one more edge. This gives us a hint to orient the whole graph G.

Since G has no cut-edges, it contains a cycle, say C. Orient C (in one of the two possible ways), so that it becomes a directed cycle C^*; obviously C^* is strongly connected. If C^* contains all the vertices of G, then we are through (after arbitrarily orienting the edges of $(E(G)-E(C))$).

If C^* does not contain all the vertices of G, extend the orientation of C^* to a larger digraph G_1 as explained above. This kind of extension can be continued until all the edges of G are oriented. □

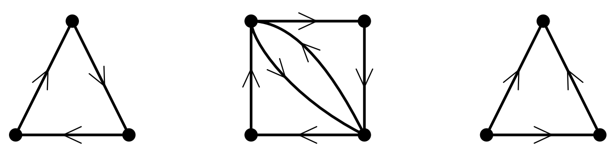
11.4 Eulerian and Hamilton digraphs

Eulerian digraphs and Hamilton digraphs are straight forward generalizations of Eulerian graphs and Hamilton graphs. Results too are analogous. However, a few results are harder to prove and even to anticipate.

• Eulerian digraphs

Definitions.

- *A directed trail in a digraph D is called an* ***Eulerian trail*** *if it contains all the arcs in D. It can be open or closed.*
- *A digraph is called an* ***Eulerian digraph*** *if it contains a closed Eulerian trail.*



D_1 D_2 D_3

Figure 11.13: D_1 and D_2 are Eulerian digraphs and D_3 is a non-Eulerian digraph.

Theorem 11.6. *A digraph D is Eulerian if and only if*

(i) D is weakly connected, and

(ii) $outdeg(v) = indeg(v)$, for every vertex v.

Proof. It is similar to the proof of Theorem 5.1 and hence it is left as an exercise. □

• Hamilton digraphs

You may recall from Chapter 6, that there are no characterizations of Hamilton graphs. It is no surprise that there are no characterizations of Hamilton digraphs either. In this section, we prove a sufficient condition for a simple digraph to contain a directed Hamilton cycle which is analogous to Dirac condition for Hamilton graphs.

Definitions.

- *A directed path in a digraph D is called a **Hamilton directed path** if it contains all the vertices of D.*
- *A directed cycle in D is called a **Hamilton directed cycle** if contains all the vertices of D.*
- *A directed graph D is called a **Hamilton digraph** if it contains a directed Hamilton cylcle.*

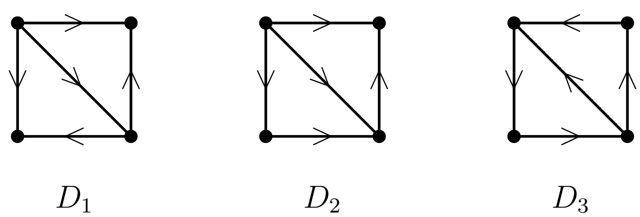


Figure 11.14: D_1 contains no directed Hamilton path. D_2 contains a directed Hamilton path but contains no directed Hamilton cycle. D_3 contains a directed Hamilton cycle and hence it is a directed Hamilton graph.

Remarks.

- D contains a directed Hamilton cycle $\Rightarrow$ D contains a directed Hamilton path.
- D contains a directed Hamilton path $\not\Rightarrow$ D contains a directed Hamilton cycle.
- D contains a directed Hamilton path $\Rightarrow$ D is unilaterally connected (Theorem 11.3.)
- D is Hamilton $\Rightarrow$ D is strong (Theorem 11.2).

Theorem 11.7 (Ghoulia-Houri, 1960). *If D is a simple digraph such that*

(i) $n \geq 3$,

(ii) $outdeg(v) \geq \frac{n}{2}$, *for every vertex* v, *and*

(iii) $indeg(v) \geq \frac{n}{2}$, *for every vertex* v,

then D *contains a directed Hamilton cycle.*

Proof. (Contradiction method) Assume that the result is false. Let C be a directed cycle in D containing maximum number of vertices. By our assumption, $V(D) - V(C) \neq \emptyset$. Let P be a directed path in $D - V(C)$ containing maximum number of vertices; let $P = P(a, b)$. Let $|V(C)| = k$ and $V(P) = p$. Fix the clock-wise direction to C; see Figure 11.15.

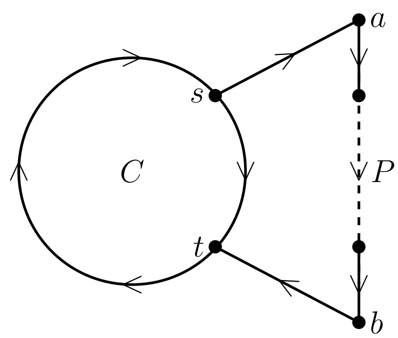


Figure 11.15: A maximum cycle C and a maximum path P in $D - V(C)$.

Clearly,

(1) $n \geq k + p$,

(2) $k > \frac{n}{2}$ (by Theorem 11.4),

(3) $N_{in}(a) \subseteq V(P - a) \cup V(C)$, by the maximality of $V(P)$.

(4) $N_{out}(b) \subseteq V(P - b) \cup V(C)$, by the maximality of $V(P)$.

Define:

$$\begin{aligned} S &= \{s \in V(C) : (s,a) \in A(D)\} \text{ and} \\ T &= \{t \in V(C) : (b,t) \in A(D)\}. \end{aligned}$$

Then

(5)

$$\begin{aligned} |S| &\geq indeg(a) - (p-1) \text{ (by (3))} \\ &\geq \tfrac{n}{2} - p + 1, \text{by the hypothesis} \\ &\geq \tfrac{n}{2} - (n-k) + 1 \text{ (by (1))} \\ &= -\tfrac{n}{2} + k + 1 \\ &\geq 1 \text{ (by (2))}. \end{aligned}$$

Similarly,

$$|T| \geq 1.$$

Since $|S| \geq 1$ and $|T| \geq 1$, we can choose $s \in S$ and $t \in T$ such that t is a successor of s on C and no internal vertex of the directed subpath $C[s,t]$ belongs to $S \cup T$.

Claim 1: There are at least p internal vertices in $C[s,t]$.

Else, the cycle

$$(s, a, P(a,b), b, t, C[t,s], s)$$

has more number of vertices than C, a contradiction to the maximality of $V(C)$.

Claim 2: If $(s,x) \in A(C)$, then $x \notin T$.

Else, we get a larger cycle as in the proof of Claim 1.

Therefore, there are at least $p + (|S| - 1)$ vertices on C which are not in T. Hence, using claims (1) and (2) we conclude that

$$\begin{aligned} k = |V(C)| &\geq p + (|S| - 1) + |T|, \\ &= (|S| + |T|) + p - 1, \\ &= 2(\tfrac{n}{2} - p + 1) + p - 1, \\ &\qquad \text{(since } |S| \geq \tfrac{n}{2} - p + 1 \text{ and } |T| \geq \tfrac{n}{2} - p + 1\text{, see (5))} \\ &= n - p + 1. \end{aligned}$$

We have thus arrived at a contradiction to (1). □

11.5 Tournaments

Tournaments form an interesting class of digraphs. They are being independently studied. An entire book of J. W. Moon (1968) is devoted to tournaments. Further survey has been done by K. B. Reid and L. W. Bineke (1978).

Definition. *An orientation of a complete graph is called a* ***tournament****.*

So in a tournament D either $(u, v) \in A(D)$ or $(v, u) \in A(D)$ (but not both), for every pair u, v of distinct vertices. A few small tournaments are shown in Figure 11.16.

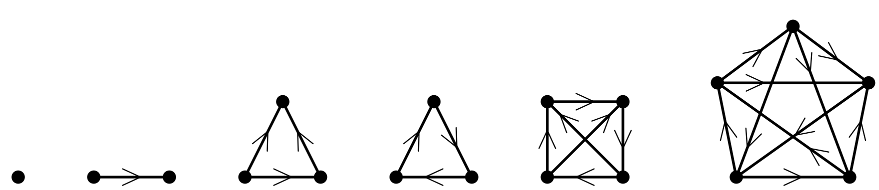


Figure 11.16: Tournaments.

Definition. *A vertex v is said to be* ***reachable*** *from a vertex u, if there is a directed path from u to v.*

Theorem 11.8. *If u is a vertex of maximum out-degree in a tournament D, then every vertex is reachable from u by a directed path of length at most 2.*

Proof. Consider the sets $N_{out}(u) = \{u_1, u_2, \ldots, u_r\}$ and $N_{in}(u) = \{v_1, v_2, \ldots, v_s\}$, where $r = outdeg(u)$ and $s = indeg(u)$. Since D is a tournament, $V(D) - \{u\} = N_{out}(u) \cup N_{in}(u)$ and $N_{out}(u) \cap N_{in}(v) = \emptyset$.

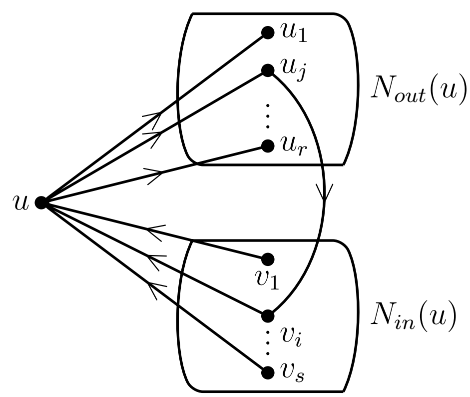


Figure 11.17

Clearly, every u_j is reachable from u by a path of length 1. We next assert that for every v_i $(1 \leq i \leq s)$, there is some u_j $(1 \leq j \leq r)$ such that (u_j, v_i) is an arc in D; so that v_i is reachable from u by a path of length 2, namely (u, u_j, v_i). Assume that our assertion is false for some v_i. Then (v_i, u_j) is an arc for every j $(1 \leq j \leq r)$. Hence,

$$N_{out}(v_i) \supseteq \{u_1, u_2, \ldots, u_r\} \cup \{u\}.$$

Thus, $outdeg(v_i) \geq outdeg(u) + 1$, a contradiction to the maximality of $outdeg(u)$. Hence, our assertion indeed holds and the theorem follows. □

Theorem 11.9. *Every tournament D contains a directed Hamilton path.*

Proof. We prove the theorem by induction on n. If $n = 1$ or 2, then the theorem is obvious. Assume that a tournament contains a directed Hamilton path if it has $n-1$ vertices and let D contain n vertices.

Let $v \in V(D)$ and consider the tournament $D - v$. By induction hypothesis, $D - v$ contains a directed Hamilton path say $(v_1, v_2, \ldots, v_{n-1})$; see Figure 11.18.

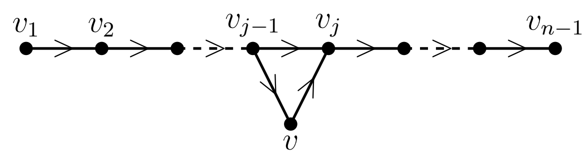


Figure 11.18: Extension of a path.

We make 3 cases and prove that D contains a directed Hamilton path in each case.

Case 1: (v, v_1) is an arc D.

Clearly, $(v, v_1, v_2, \ldots, v_{n-1})$ is a directed Hamilton path in D.

Case 2: (v, v_1) is not an arc (and hence (v_1, v) is an arc) but there is some i $(2 \leq i \leq n-1)$ such that (v, v_i) is an arc.

Let j $(2 \leq j \leq n-1)$ be the smallest integer such that (v, v_j) is an arc in D. This means that (v, v_{j-1}) is not an arc; and hence (v_{j-1}, v) is an arc. But then

$$(v_1, v_2, \ldots, v_{j-1}, v, v_j, v_{j+1}, \ldots, v_{n-1})$$

is a directed Hamilton path in D.

Case 3: There is no i $(1 \leq i \leq n-1)$, such that (v, v_i) is an arc.
This means, in particular, (v_{n-1}, v) is an arc in D. But then

$$(v_1, v_2, \ldots, v_{n-1}, v)$$

is a directed Hamilton path in D. □

Corollary. *Every tournament is unilaterally connected.*

Proof. A consequence of Theorem 11.3. □

A tournament need not contain a directed Hamilton cycle. In Figure 11.16, the second and third tournaments are non-Hamilton whereas the others are Hamilton. However, every strong tournament on at least three vertices, contains a directed Hamilton cycle. We shall prove a stronger assertion.

Theorem 11.10. *Every strong tournament D on n (≥ 3) vertices contains a directed cycle of length k, for every k, $3 \leq k \leq n$.*

Proof. We first prove that D contains a directed 3-cycle and next show that, if D contains a directed k-cycle, for some k $(3 \leq k \leq n-1)$, then it contains a directed $(k+1)$-cycle; so that D contains a directed k-cycle, for every k, $3 \leq k \leq n$.

(1) D contains a directed 3-cycle.

Let $v \in V(D)$ and consider $N_{out}(v)$ and $N_{in}(v)$; see Figure 11.19.

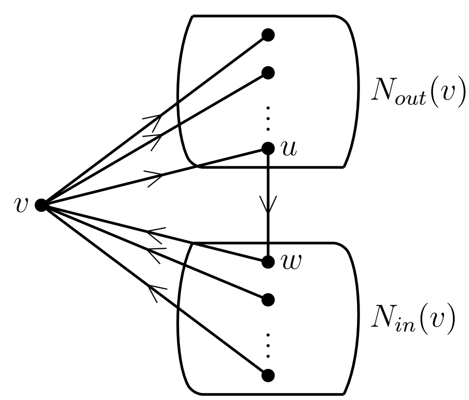


Figure 11.19: A 3-cycle in a strong tournament.

Since D is strongly connected, $N_{out}(v)$ and $N_{in}(v)$ are non-empty. Moreover, there is an arc (u, w) in D where $u \in N_{out}(v)$ and $w \in N_{in}(v)$ because every directed path from v to a vertex $w \in N_{in}(v)$ must pass through a vertex in $N_{out}(v)$. But then (v, u, w, v) is a directed cycles of length 3.

(2) Let $C(v_1, v_2, \ldots, v_k, v_1)$ be a directed cycle of length k in D, for some k $(3 \leq k \leq n-1)$.

We make two cases and in each case construct a directed $(k+1)$-cycle.

Case 1: There is a vertex $u \in D - V(C)$ such that (v_i, u) and (u, v_j) are arcs in D for some i and j; without loss of generality, let $(v_1, u) \in A(D)$.

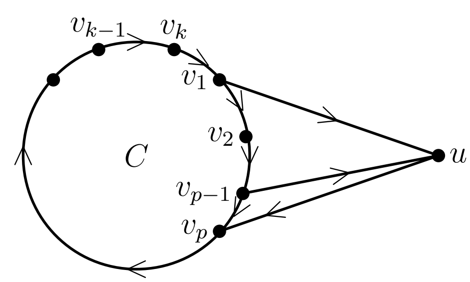


Figure 11.20: An extension of a k-cycle to a $(k+1)$-cycle.

Let p $(2 \leq p \leq k)$ be the smallest integer such that $(u, v_p) \in A(D)$. So, $(v_{p-1}, u) \in A$. Then

$$(v_1, v_2, \ldots, v_{p-1}, u, v_p, v_{p+1}, \ldots, v_k, v_1)$$

is directed cycle of length $(k+1)$ in D; see Figure 11.20.

If the assumption made in Case 1 does not hold, then the following must hold.

Case 2: If $u \in D - V(C)$, then one of the following holds.

(a) $(u, v_i) \in A(D)$, for every i, $1 \leq i \leq k$,

(b) $(v_i, u) \in A(D)$, for every i, $1 \leq i \leq k$.

Define
$X = \{u \in D - V(C); (v_i, u) \in A(D)$, for every $i, 1 \le i \le k\}$, and
$Y = \{w \in D - V(C); (w, v_i) \in A(D)$, for every $i, 1 \le i \le k\}$.

Since D is strong and since for every $u \in D - V(C)$ either (a) or (b) holds, it follows that $X \neq \emptyset$, $Y \neq \emptyset$, $X \cap Y = \phi$ and $X \cup Y = V(D) - V(C)$.

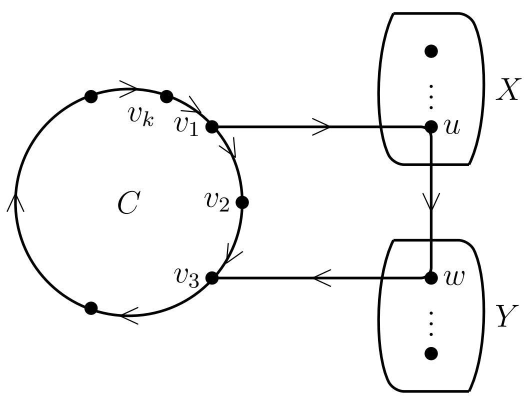


Figure 11.21

There are vertices $u \in X$ and $w \in Y$ such that (u, w) is an arc in D, because every directed path from a vertex in C to a vertex in Y must pass through a vertex in X. Now, since $u \in X$ and $w \in Y$, (v_1, u) and (w, v_3) are arcs in D. But then $(v_1, u, w, v_3, v_4, \ldots, v_k, v_1)$ a directed cycle of length $k + 1$ in D. □

Corollary. *A tournament is strong if and only if it contains a directed Hamilton cycle.*

Proof. A consequence of Theorems 11.2 and 11.10. □

Exercises

1. Draw all the non-isomorphic strong simple digraphs on 4 vertices and 5 arcs.

2. If D is a digraph in which every vertex has out-degree 1, then show that D has exactly one directed cycle.

3. Draw (as many as you can) simple non-isomorphic digraphs on 7 vertices in which every vertex has out-degree 1 and every vertex has in-degree 1. (Do you recognize any relation on such digraphs and partition of 7?).

4. Let G be a k-regular graph on vertices $v_1, v_2, \ldots, v_n$ and let D be an orientation of G. Show that $\sum_{i=1}^{n}(outdeg(v_i))^2 = \sum_{i=1}^{n}(indeg(v_i))^2$.

5. A simple digraph D is called ***k*-regular** if $outdeg(v) = indeg(v) = k$, for every vertex $v \in V(D)$.
   (a) Draw a 2-regular digraph on 5 vertices.
   (b) Let k and n be integers such that $0 \leq k < n$. Describe a construction to obtain a k-regular digraph on n vertices.

6. Prove or disprove: For every $n \geq 1$, there is a simple digraph on n vertices in which every vertex has odd out-degree.

7. (a) Construct pairs (D_1, D_2) of simple digraphs on n vertices for $n = 2, 3, 4$ where $V(D_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_n\}$ such that
      i. $D_1 \not\simeq D_2$.
      ii. $D_1 - u_i \simeq D_2 - v_i$, $i = 1, 2, \ldots, n$.
   (b) Verify that the digraphs D_1 and D_2 shown below have the properties (i) and (ii) stated above.

8. If D is a weakly connected digraph, then show that $m \geq n - 1$.

9. If D is a digraph with $m \geq (n-1)(n-2) + 1$, then show that D is weakly connected.

10. If D is a strongly connected simple digraph, then show that $n \leq m \leq n(n-1)$.

11. If D is a simple digraph with $m \geq (n-1)^2 + 1$, then show that D is strongly connected.

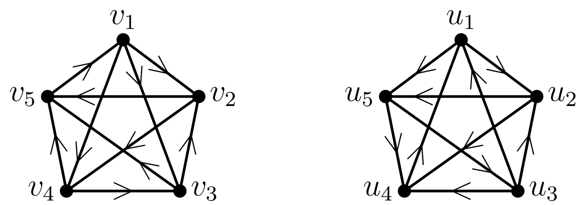


Figure 11.22

12. Give an example of a simple digraph with the following properties:

    (a) $m = (n-1)(n-2)$, which is disconnected;

    (b) $m = n$, which is strongly connected;

    (c) $m = n-1$, which is weakly connected;

    (d) $m = n-1$, which is unilaterally connected;

    (e) $m = (n-1)^2$, which is not strongly connected.

    In view of the existence of these examples, discuss the merits of the bounds stated in Exercises 8 to 11.

13. Show that a digraph is strong if and only if its converse digraph is strong. (The ***converse digraph*** $\overleftarrow{D}$ of D has vertex set $V(\overleftarrow{D}) = V(D)$. $(u,v) \in A(\overleftarrow{D})$ iff $(v,u) \in A(D)$.)

14. Use the proof of Theorem 11.5 to give a strong orientation to the following graphs.

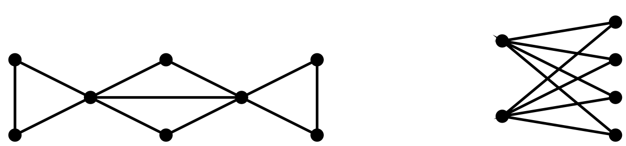


Figure 11.23

15. If D is a weakly connected simple digraph such that $D - v$ is strong for some vertex $v \in V(D)$, then show that D is unilaterally connected.

16. The adjacency matrix $M = [a_{ij}]$ of a digraph D on n vertices $v_1, v_2, \ldots, v_n$ is a $n \times n$ matrix where $a_{ij} = 1$ if (v_i, v_j) is an arc in D and $a_{ij} = 0$, otherwise. Show that

    (a) $\sum_{j=1}^{n} a_{ij} = outdeg(v_i)$, $t = 1, 2, \ldots, n$;

    (b) $\sum_{j=1}^{n} a_{ij} = indeg(v_i)$, $t = 1, 2, \ldots, n$;

    (c) The (i, j)-th entry $[M^p]_{ij}$ in M^p is the number of directed walks of length p from v_i to v_j.

17. Every graph G has an orientation D such that $|outdeg(v) - indeg(v)| \leq 1$, for every vertex v.

18. Let D be a digraph and let $r = \max\limits_{v \in V}\{outdeg(v), indeg(v)\}$. Prove that there is an r-regular digraph H such that D is a subdigraph of H.

19. Prove that there exist regular tournaments of every odd order but there are no regular tournaments of even order.

20. Show that in a tournament there is at most one vertex of out-degree zero and at most one vertex of in-degree zero.

21. Let D be a strong tournament. Given any k, $1 \leq k \leq n - 3$, show that there exists a set $S \subseteq V(D)$ of k vertices such that $D - S$ is strongly connected.

22. (a) Draw a tournament on 5 vertices in which every vertex has the same out-degree.

    (b) If a tournament has n vertices and every vertex has out-degree d, find d.

23. Draw:

    (a) A unilaterally connected (but not strongly connected) tournament on 5 vertices.

    (b) A strongly connected tournament on 5 vertices.
    Justify that your tournaments indeed have the required properties.

24. Show that a tournament D is not strongly connected if and only if there is a partition of $V(D)$ into two subsets A and B such that every arc in between A and B is of the form (u, v), where $u \in A$ and $v \in B$.

25. A tournament D is called a ***transitive tournament*** if $(u, w) \in A(D)$ whenever (u, v) and $(v, w) \in A(D)$. Show that a tournament is transitive if and only if the vertices of D can be ordered $v_1, v_2, \ldots, v_n$ such that $outdeg(v_1) = 0$, $outdeg(v_2) = 1, \ldots, outdeg(v_n) = n - 1$.

26. Show that a tournament is transitive if and only if it does not contain any directed cycle.

27. If T is a tournament on n vertices $v_1, v_2, \ldots, v_n$ with $outdeg(v_i) = s_i,\ i = 1, 2, \ldots, n$ then show that

    (a) $\sum_{i=1}^{n} s_i \geq \dfrac{k(k-1)}{2}\ 1 \leq k < n.$

    (b) $\sum_{i=1}^{n} s_i = \dfrac{n(n-1)}{2};$

    (c) $\sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} (n - 1 - s_i)^2;$

28. A vertex v in a tournament T is called a winner if every vertex can be reached from v by a directed path of length ≤ 2. Show the following:

    (a) No tournament has exactly two winners.

    (b) For every $n \neq 2, 4$, there is a tournament of order n in which every vertex is a winner.