MODULE VI

REFERENCE:

DISCRETE MATHEMATICAL STRUCTURES WITH APPLICATIONS TO COMPUTER SCIENCE

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Predicates

In statements involving variables, there are two parts –

- variable (is the subject of the statement)
- *predicate* (refers to a property that the subject can have).

Consider a sentence: x is greater than 2. Here “is greater than 2” is the predicate and x is the subject or variable.

- If values are assigned to all the variables, the resulting sentence is a proposition.

e.g. $x < 9$ is a predicate

$4 < 9$ is a proposition

Propositional Function

- A propositional function (or an open sentence) defined on A is a predicate together with subjects.

- It is denoted by the expression P(x) which has the property that P(a) is true or false for each $a \in A$.

- The set A is called domain of P(x) and the set Tp of all elements of A for which P (a) is true is called the truth set of P(x).

Universe of Discourse

- Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the ***universe of discourse***.

Quantifiers

- Quantification is the way by which a Propositional function can turn out to be a proposition.

- The expressions '**f o r all**' and '**there exists**' are called quantifiers.

- The process of applying quantifier to a variable is called quantification of variables.

Universal quantification

- The universal quantification of a predicate P(*x*) is the statement, "For all values of *x*, P(*x*) is true."

- The universal quantification of P(*x*) is denoted by

  ʋ for all x P(*x*).

- The symbol ʋ is called the universal quantifier.

- Universal quantification can also be stated in English as "for every *x*", "every *x*", or "for any *x*."

Example:

- The sentence P(x) : $-(-x) = x$ is a predicate that makes sense for real numbers x. The universal quantification of P(x), $\forall\ x$ P(x) is a true statement because for all real numbers, $-(-x) = x$.

- 2) Let Q(x) : $x + 1 < 5$, then $\forall$ Q(x) : $x + 1 < 5$ is a false statement, as Q(5) is not true.

Existential quantification

- The existential quantification of a predicate P(x) is the statement

  "There exists a value of x for which P(x) is true."

- The existential quantification of P (x) is denoted $\exists xP(x)$.

- The symbol $\exists$ is called the existential quantifier.

Example:

- Let $Q : x + 1 < 4$.

The existential quantification of Q(x), $\exists\, x\, Q(x)$ is a true statement, because Q(2) is true statement.

- The statement $\exists\, y$, y + 2 = y is false. There is no value of y for which the propositional function y+2 = y produces a true statement.

Negation of Quantified statement

- Negations

The *negation* of a universal quantification is an existential quantification.

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$$

- The negation of an existential quantification is a universal quantification.

$$\neg \exists x Q(x) \Leftrightarrow \forall x \neg Q(x)$$

- For example,

The negation of all men are mortal is: There is a man who is not mortal.

Statement	When True?	When False?
$\forall x\ P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\ P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

- **Example :**

  Express the statement using quantifiers:

  "Every student in your school has a computer or has a friend who has a computer."

Solution :

Let c(x) : "x has a computer"

F(x,y) : "x and y are friends"

Thus, We have

$$\forall x\,(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

- Every student in this class has visited either Canada or Mexico.

  Let $M(x)$ be the statement "x has visited Mexico" and $C(x)$ be the statement "x has visited Canada".

$\forall x(C(x) \vee M(x))$, the universe of discourse for x is the set of all the students in this class.

- If somebody is female and is a parent, then this person is someone's mother

Let $F(x)$ be the statement "x is female",
$P(x)$ be the statement "x is a parent", and
$M(x, y)$ be the statement "x is the mother of y".

$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$, the universe of discourse for x and y is the set of all people.

THEORY OF INFERENCE FOR THE PREDICAT E CALCULAS

Rule of Inference	Name
$\forall x P(x)$ $\therefore P(c)$ if $c \in U$	Universal instantiation
$P(c)$ for an arbitrary $c \in U$ $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element $c \in U$	Existential instantiation
$P(c)$ for some element $c \in U$ $\therefore \exists x P(x)$	Existential generalization

Table 1-6.1

$(\exists x)(A(x) \lor B(x)) \Leftrightarrow (\exists x)A(x) \lor (\exists x)B(x)$	E_{23}
$(x)(A(x) \land B(x)) \Leftrightarrow (x)A(x) \land (x)B(x)$	E_{24}
$\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$	E_{25}
$\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$	E_{26}
$(x)A(x) \lor (x)B(x) \Rightarrow (x)(A(x) \lor B(x))$	I_{15}
$(\exists x)(A(x) \land B(x)) \Rightarrow (\exists x)A(x) \land (\exists x)B(x)$	I_{16}

Table 1-6.2

$(x)(A \lor B(x)) \Leftrightarrow A \lor (x)B(x)$	E_{27}
$(\exists x)(A \land B(x)) \Leftrightarrow A \land (\exists x)B(x)$	E_{28}
$(x)A(x) \rightarrow B \Leftrightarrow (\exists x)(A(x) \rightarrow B)$	E_{29}
$(\exists x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$	E_{30}
$A \rightarrow (x)B(x) \Leftrightarrow (x)(A \rightarrow B(x))$	E_{31}
$A \rightarrow (\exists x)B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$	E_{32}

Valid Argument

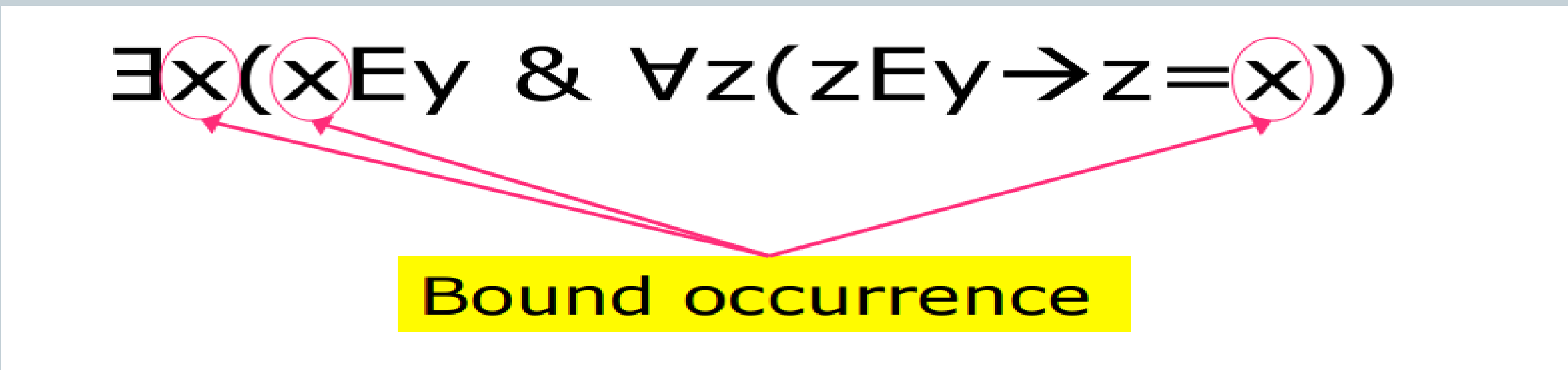
- An argument in propositional logic is a sequence of compound propositions - involving propositional variables
- All propositions in the argument are called **hypothesis** or **Premises**. The final proposition is called the **conclusion**.
- An argument form is valid if no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.
- Thus we say the conclusion C can be drawn from a given set of premises or the argument is valid if the conjunction of all the premises implies the conclusion is a tautology.

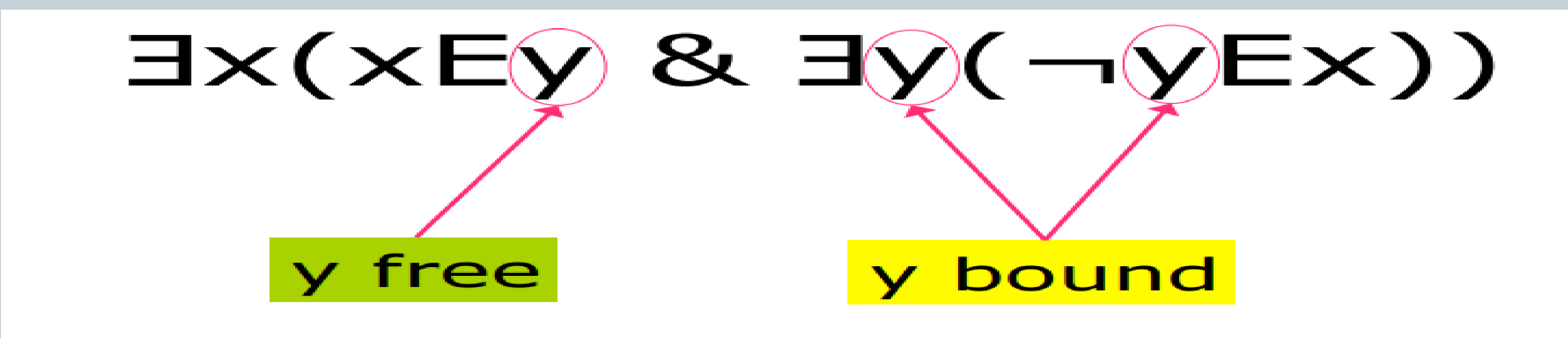
Free and Bound Variables

- An expression like P (x) is said to have a free variable x (meaning, x is undefined).

- A quantifier (either ∀ or ∃) operates on an expression having one or more free variables, and binds one or more of those variables, to produce an expression having one or more bound variables.

Ex. ∃ x [x + y = z], x is bound but y and z are free variables.





- $\forall x\ \exists x\ P(x)$ - x is not a free variable in $\exists x\ P(x)$, therefore the $\forall x$ binding isn't used.
- $(\forall x\ P(x)) \wedge Q(x)$ - The variable x in $Q(x)$ is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a proposition!

  $\forall x\ P(x) \wedge Q(x) \neq \forall x\ (P(x) \wedge Q(x))$

  $(\forall x\ P(x)) \wedge Q(x)$

  $\forall x\ P(x) \wedge Q(y)$: clearer notation
- $(\forall x\ P(x)) \wedge (\exists x\ Q(x))$ – This is legal, because there are 2 different x's!

Let $F(x, y)$ be the statement "x *loves* y," where the universe of discourse for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry. $(\forall x)\ F(x, Jerry)$

b) Everybody loves somebody. $(\forall x)(\exists y)\ F(x,y)$

c) There is somebody whom everybody loves. $(\exists y)\ (\forall x)\ F(x,y)$

d) Nobody loves everybody. $\neg(\exists x)(\forall y)\ F(x,y)$

e) There is somebody whom Lydia does not love. $(\exists x)\ \neg F(Lydia,x)$

f) There is somebody whom no one loves. $(\exists x)(\forall y)\ \neg F(y,x)$

g) There is exactly one person whom everybody loves. $(\exists! x)(\forall y)\ F(y,x)$

h) There are exactly two people whom Lynn loves.

$$(\exists x)\ (\exists y)\ ((x \neq y) \wedge F(Lynn,x) \wedge F(Lynn,y) \wedge (\forall z)\ (F(Lynn,z) \rightarrow (z=x) \vee (z=y)))$$

i) Everyone loves himself or herself $(\forall x)\ F(x,x)$

j) There is someone who loves no one besides himself or herself.

$$(\exists x)\ (\forall y)\ (F(x,y) \leftrightarrow x=y)$$

To be continued……