Matrix Eigen Value Problems

Consider the vector equation,

$$Ax = \lambda x \quad — (1)$$

where A is a given square matrix, λ is an unknown constant and x is an unknown vector.
The MEV Problem is to find λ and x that satisfies (1).
The value of λ that satisfy (1) is called Eigen Value of A and corresponding value of x is called eigen vector of A.

How to find Eigen Value and Eigen Vector

Consider the vector equation,

$$Ax = \lambda x$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = \lambda x_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = \lambda x_2$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = \lambda x_n$$

ie $(a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$
$$a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n = 0$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n = 0$$

In matrix notation, this can be written as.

$$[A - \lambda I]x = 0$$

where I is the identity matrix with same order of A.
The matrix $A - \lambda I$ is called characteristic matrix.
$D(\lambda) = |A - \lambda I|$ is called characteristic determinant.
The equation $|A - \lambda I| = 0$ is called characteristic equ: of A.
By expanding characteristic equ: we get a polynomial of degree n and that polynomial is called characteristic polynomial of A.

The eigen values of square matrix A are the roots of characteristic equ: of A.
Hence $n \times n$ matrix has atleast one eigen value and almost n eigen values.

4) $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$$[A-\lambda I] = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5-\lambda & 2 \\ 2 & (-2-\lambda) \end{bmatrix}$$

$|A-\lambda I| = 0$

$(-5-\lambda)(-2-\lambda)-4$

$= 10+5\lambda+2\lambda+\lambda^2-4$

$0 = \lambda^2+7\lambda+6$

$\lambda = -1, -6$ are the eigen values.

Now consider $[A-\lambda I]x$

$$= \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$[A-\lambda I]x = 0$

$(-5-\lambda)x_1 + 2x_2 = 0$

$2x_1 - (2+\lambda)x_2 = 0$

$\lambda = -1$

$-4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0$

$$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \rightarrow 2R_2 + R_1$$

$-4x_1 + 2x_2 = 0$

$x_2 - 2x_1 = 0$

$x_2 = 2x_1$

$(n-R)=1$

Put $x_1=1$
$x_2=2$ $X=\begin{bmatrix}1\\2\end{bmatrix}$

$\lambda=-6$

$2x_1+4x_2=0$

$x_1+2x_2=0$ — (1)

$x_1+2x_2=0$ — (2)

$$\begin{bmatrix}1 & 2 & 0\\1 & 2 & 0\end{bmatrix} \quad R_2 \rightarrow R_2-R_1$$

$$\begin{bmatrix}1 & 2 & 0\\0 & 0 & 0\end{bmatrix}$$

$x_1+2x_2=0$

$x_1=-2x_2$

$R_A=1$ $R_{\tilde{A}}=1$

$n=2$

$(n-R)=1$

Put $x_2=1$

$x_1=-2$

$$X=\begin{bmatrix}-2\\1\end{bmatrix}$$

2) $A=\begin{bmatrix}0 & 4\\-4 & 0\end{bmatrix}$

$$[A-\lambda I]=\begin{bmatrix}-\lambda & 4\\-4 & -\lambda\end{bmatrix}$$

$|A-\lambda I|=0$

$\lambda^2+16=0$

$\lambda=\pm 4i$

$\lambda=+4i,-4i$ are eigen values

$(A-\lambda I)X=0$

$$\begin{bmatrix} & \\ -4 & -\lambda \end{bmatrix} \begin{bmatrix} \\ x_2 \end{bmatrix} = 0$$

$$-\lambda x_1 + 4x_2 = 0$$
$$-4x_1 - \lambda x_2 = 0$$

$$-\lambda x_1 + 4x_2 = 0$$
$$-4x_1 - \lambda x_2 = 0$$

$$\lambda = 4i$$

$$-4ix_1 + 4x_2 = 0$$
$$ix_1 + x_2 = 0 \longrightarrow (1)$$
$$-4x_1 - 4ix_2 = 0$$
$$x_1 + ix_2 = 0 \Longrightarrow (2)$$

$$\begin{bmatrix} -i & 1 & 0 \\ 1 & i & 0 \end{bmatrix} \quad R_2 \longrightarrow iR_2 + R_1$$

$$\begin{bmatrix} -i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-ix_1 + x_2 = 0$$
$$x_2 = ix_1$$
$$x_1 = 1$$
$$x_2 = i$$
$$X = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda = -4i$$

$$4ix + 4x_2 = 0$$
$$ix_1 + x_2 = 0 \quad — (1)$$
$$-4x_1 + 4ix_2 = 0$$
$$-x_1 + ix_2 = 0 \quad — (2)$$

$$\begin{bmatrix} i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} \quad R_2 \longrightarrow iR_2 + R_1$$

$$\begin{bmatrix} i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$ix_1 = -x_2 \qquad x_2 = 1 \qquad x_1 = \frac{-1}{i}$$

$$X = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix}$$

$$4-\lambda \left((5-\lambda)(3-\lambda)\right] - 2\left[(6-2\lambda) - 2\left[10-2\lambda\right)\right]$$

$$4-\lambda\left[15-5\lambda-3\lambda+\lambda^2\right] - 12+4\lambda \quad -20+4\lambda$$

$$(4-\lambda)(\lambda^2-8\lambda+15)+(8\lambda-32)$$

$$(4-\lambda)(\lambda^2-8\lambda+15)-8(4-\lambda)$$

$$(4-\lambda)(\lambda^2-8\lambda+7)=0$$

$\lambda = 4$, $\lambda = 7, 1$ are eigen values..

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(4-\lambda)x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 + (5-\lambda)x_2 = 0$$

$$-2x_1 + (3-\lambda)x_3 = 0$$

$\lambda = 4$

$$x_2 - x_3 = 0$$

$$2x_1 - x_2 = 0$$

$$-2x_1 - x_3 = 0$$

$$\begin{bmatrix} 0 & 2 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix}$$

$R_3 \longleftrightarrow R_2$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_2 \longrightarrow R_2 + R_1$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 2R_2 \longrightarrow \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$2x_1 + x_2 = 0$

$x_1 = 1 \qquad x_2 = -2$

$x_2 - x_3 = 0$

$-2 - x_3 = 0 \qquad \underline{x_3 = -2}$

$$X = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$\lambda = 1$

$3x_1 + 2x_2 - 2x_3 = 0$

$2x_1 + 4x_2 = 0$

$-2x_1 + 2x_3 = 0$

$$A_1 = \left[\begin{array}{ccc|c} 3 & 2 & -2 & 0 \\ 2 & 4 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right]$$

$R_2 \longrightarrow 3R_2 - 2R_1 \Longrightarrow \begin{bmatrix} 3 & 2 & -2 & 0 \\ 0 & 8 & 4 & 0 \\ -2 & 0 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 6 & 4 & 2 & \vert & 0 \\ 0 & 0 & 0 & \vert & 0 \end{bmatrix}$$

$R < n$

$(n-R) = 1$

$3x_1 + 2x_2 - 3x_3 = 0$

$4x_2 + 2x_3 = 0$

ie $4x_2 = -2x_3$

Put $x_2 = 1$ $\quad x_3 = -2$

$3x_1 + 2x_2 - 3x_3 = 0$

$3x_1 = -2\times 1 + 3x - 2 = -2 - 6$

$x_1 = -8/3$

$$X_2 = \begin{bmatrix} -8/3 \\ 1 \\ -2 \end{bmatrix}$$

$\lambda = 7$

$-3x_1 + 2x_2 - 2x_3 = 0$

$2x_1 - 2x_2 + 0 = 0$

$-2x_1 - 4x_3 = 0$

$$A_2 = \begin{bmatrix} -3 & 2 & -2 & 0 \\ 2 & -2 & 0 & 0 \\ -2 & 0 & -4 & 0 \end{bmatrix}$$

A matrix A is called Symmetric if $A = A^T$ and A is called Skew Symmetric if $A = -A^T$.

Consider the matrix $A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$

$$A^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$A = A^T$ ∴ matrix is Symmetric

Consider the matrix $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

$A = -A^T$

∴ A is skew Symmetric.

Orthogonal Matrices

A matrix A is called orthogonal if $A^T = A^{-1}$. To find inverse of matrix A, if A is a 2x2 matrix, then,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then $A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

we can find inverse of A using row transformations.
for that consider the matrix.

$$[A \mid I]$$

where I is an identity matrix of same order.
Then using row transformations convert matrix A into an identity matrix.
The new form of I will give A^{-1}.

$$\text{ie } [I \mid A^{-1}]$$

Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right]$$

$R_2 \rightarrow -R_2$

$R_3 \rightarrow -R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = A^T$$

$\therefore$ Given matrix is Orthogonal.

The set of all eigen values of A is called Spectrum of A.

2) Check whether the matrix A is Symmetric, Skew symmetric or Orthogonal. find the Spectrum of A and find eigen values ?

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$= 1/5 \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{25} & 4/25 \\ -4/25 & 3/25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{125} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$$A \neq A^T$$

$$A \neq -A^T$$

$$A^T = A^{-1}$$

$\therefore$ it is orthogonal

$$\begin{bmatrix} 4 & 3-\lambda \end{bmatrix}$$

$|A-\lambda I| = 0$

$(3-\lambda)^2 + 16 = 0$

$25 + \lambda^2 - 6\lambda = 0$

$\lambda = 3+4i,\ 3-4i$ are the eigen values.

$\{3+4i, 3-4i\}$ = Spectrum

$[A-\lambda I]x = 0$

$$\begin{bmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$(3-\lambda)x_1 - 4x_2 = 0$

$4x_1 + (3-\lambda)x_2 = 0$

Put $\lambda = 3-4i$

$4i\, x_1 - 4x_2 = 0$

$i x_1 - x_2 = 0$ — (i)

$4x_1 + 4i\, x_2 = 0$

$x_1 + i x_2 = 0$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right]$$

$R_2 \rightarrow iR_2 - R_1$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$i x_1 - x_2 = 0$

$i x_1 = x_2$

$x_1 = 1$

$x_2 = i$

Put $\lambda = 3+4i$

$-i x_1 - x_2 = 0$

$i x_1 + x_2 = 0$ — ①

$x_1 - i x_2 = 0$

$$\left[\begin{array}{cc|c} 4i & 4 & 0 \\ 4 & -4i & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 i - R_1$

$$\left[\begin{array}{cc|c} 4i & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$i x_1 = x_2$

$x_1 = 1$

$x_2 = i$

$$X = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Eigen Vectors are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Diagonalisation of Matrix

A matrix A is called diagonalisable if there exist a matrix P such that $P^{-1}AP$ is a diagonal matrix.

Theorem

If a square matrix A has a basis of eigen vectors then $D = X^{-1}AX$ is a diagonal matrix with eigen values of A as the entries on the main diagonal. Here X is the matrix with the eigen vectors as column vectors.

1) Find an Eigen basis and diagonalise.

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$4 + \lambda^2 - 4\lambda - 16 = 0$$

$$\lambda^2 - 4\lambda - 12 = 0$$

$\lambda = 6, -2$ are eigen values

$$(A - \lambda I)X = 0$$

$$(2-\lambda)x_1 + 4x_2 = 0$$

$$4x_1 + (2-\lambda)x_2 = 0$$

at $\lambda = -2$

$$4x_1 + 4x_2 = 0$$

$$4x_1 + 4x_2 = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = -x_1$$

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Put $\lambda = 6$

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$X^{-1}A = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$$

1) $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\begin{bmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{bmatrix} = 0$$

$-304 + 19\lambda - 16\lambda + \lambda^2 + 294$

$0 = \lambda^2 + 3\lambda - 10$

$\lambda = -5, 2$

$(-19-\lambda)x_1 + 7x_2 = 0$

$-42x_1 + (16-\lambda)x_2 = 0$

$\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & -1 & 0 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$-1-\lambda\,(-2+\lambda-2\lambda+\lambda^2)+1\,(\lambda-2)=0$$

$$2-\lambda+2\lambda-\lambda^2+2\lambda-\lambda^2+2\lambda^2-\lambda^3+\lambda-2=0$$

$$4\lambda-\lambda^3=0$$

$$\lambda^3-4\lambda$$

$$\lambda(\lambda^2-4)=0$$

$$\lambda=2,-2,0$$

$$|A-\lambda I|x=0$$

$$\begin{bmatrix} -1-\lambda & -1 & 0 \\ -1 & -1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$(-1-\lambda)x_1-x_2=0$$

$$-x_1\cdot(-1-\lambda)x_2=0$$

$$(2-\lambda)x_3=0$$

$\lambda=-2$

$$x_1-x_2=0$$

$$-x_1+x_2=0$$

$$4x_3=0$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2+R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$4x_3 = 0$

$x_3 = 0$

$x_1 - x_2 = 0$

$x_1 = x_2$

Put $x_1 = 1$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Put $\lambda = 2$

$$\begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$-3x_1 - x_2 = 0$ $\quad$ $-3x_1 = x_2$

$-x_2 = 0$ $\quad$ $x_1 = x_2 = 0$

Put $x_3 = 1$

$$\therefore X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Put $\lambda = 0$

$-x_1 - x_2 = 0$

$-x_1 - x_2 = 0$

$2x_3 = 0$

$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 - R_1$$

$-x_1 - x_2 = 0$

$x_1 = -x_2$

$x_3 = 0$

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 + \frac{1}{2} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$R_3 \to -\frac{1}{2} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$X^{-1} A \cdot$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= [-2, 2, 0]$$

Similar Matrices

An $n \times n$ matrix $\hat{A}$ is called similar to an $n \times n$ matrix A if $\hat{A} = P^{-1}AP$ for same non-singular $n \times n$ matrix P. If $\hat{A}$ is similar to A, then $\hat{A}$ has the same Eigen values of A. If X is an eigen vector of A, then $Y = P^{-1}X$ is an eigen vector of $\hat{A}$.

1) find $\hat{A}$ and show that, if y is an eigen vector of $\hat{A}$, then $X = PY$ is an eigen vector of A.

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -3 \\ -2 & -4 \end{bmatrix}$$

$$P^{-1}A = \frac{1}{-2} \begin{bmatrix} -15 & 5 \\ -22 & 4 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{-2} \begin{bmatrix} 15 & 5 \\ -22 & 4 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$$

$$|\hat{A} - \lambda I| = 0$$

$$\begin{vmatrix} -25-\lambda & 12 \\ -50 & 25-\lambda \end{vmatrix} = 0$$

$$(-25-\lambda)(25-\lambda) + 50 \times 12 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda = \pm 5$$

$\lambda = -5$

$$(\hat{A} - \lambda I)x = 0$$

$$(-25-\lambda)x_1 + 12x_2 = 0$$

$$-50x_1 + (25-\lambda)x_2 = 0$$

$$-20x_1 + 12x_2 = 0$$

$$-50x_1 + 30x_2 = 0$$

$$\left[\begin{array}{cc|c} -10 & 6 & 0 \\ -5 & 3 & 0 \end{array}\right]$$

$$x_2 \rightarrow 2x_2 - x_1$$

$$\left[\begin{array}{cc|c} -10 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-10x_1 + 6x_2 = 0$$

$$x_1 = \frac{6x_2}{10}$$

$$X = \begin{bmatrix} 1 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$-30x_1 + 12x_2 = 0$$

$$-50x_1 + 20x_2 = 0$$

$$\begin{bmatrix} -30 & 12 & 0 \\ -50 & 20 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2/10$$

$$\begin{bmatrix} \frac{-30}{5} & 12 & 0 \\ & 2 & 0 \end{bmatrix}$$

$$R_2 \rightarrow 6R_2 - R_1$$

$$\begin{bmatrix} -30 & 12 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$30x_1 + 12x_2 = 0$$

$$X = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$|A-\lambda I|=0$$

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}$$

$\lambda = -5$

$[A-\lambda I]x = 0$

$(3-\lambda)x_1 + 4x_2 = 0$

$4x_1 + (-3-\lambda)x_2 = 0$

$8x_1 + 4x_2 = 0$

$2x_1 + x_2 = 0$ —— (1)

$4x_1 + 2x_2 = 0$

$2x_1 + x_2 = 0$ —— (2)

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$2x_1 = -x_2$

$x_1 = 1$

$$X = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$-2x_1 + 4x_2 = 0$

$4x_1 - 8x_2 = 0$

$$\begin{bmatrix} -2 & 4 & 0 \\ 4 & -8 & 0 \end{bmatrix}$$

$R_2 \rightarrow 2R_1 + R_2$

$$\begin{bmatrix} -2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-2x_1 + 4x_2 = 0$

$x_1 = 2x_2$

$$X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

y - eigenvector of $\hat{A}$

$$y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$X = Py$

$$= \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

at $\lambda = -5$

$$y = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

3) $A = \begin{bmatrix} -2 & 0 & 12 \\ -2 & 4 & 4 \\ 2 & 0 & 12 \end{bmatrix}$ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 12 \\ -2 & 4 & 4 \\ -2 & 0 & 12 \end{bmatrix}$$

$$P^{-1}AP \begin{bmatrix} -2 & 4 & 4 \\ -2 & 0 & 12 \\ -2 & 0 & 12 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 4 & -2 & 4 \\ 0 & -2 & 12 \\ 0 & -2 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 4-\lambda & -2 & 4 \\ 0 & -2-\lambda & 12 \\ 0 & -2 & 12-\lambda \end{bmatrix}$$

$$4-\lambda\left[(-2-\lambda)(12-\lambda)+24\right]=0$$

$$4-\lambda=0$$

$$\lambda=4$$

$$-24+2\lambda-12\lambda+\lambda^2+24=0$$

$$-10\lambda+\lambda^2=0$$

$$\lambda(\lambda-10)=0$$

$$\lambda=\underline{10,0,4}$$

at $\lambda=0$

$$\begin{bmatrix} 4 & -2 & 4 \\ 0 & -2 & 12 \\ 0 & -2 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$4x_1-2x_2+4x_3=0$$

$$2x_1-x_2+2x_3=0 \quad \text{—— (1)}$$

$$-2x_2+12x_3=0$$

$$-x_2+6x_3=0 \quad \text{—— (2)}$$

$$-x_2+6x_3=0$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right]$$

$$R_3 \rightarrow R_3-R_2$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-x_2+6x_3=0$$

$$6x_3=x_2$$

$$x_3=x_2/6$$

$$2x_1-x_2+2x_3=0$$

$$2x_1-x_2+\frac{x_2}{3}=2x_1-\frac{2}{3}x_2=0$$

$$2x_1=\frac{2}{3}x_2 \Rightarrow x_1=-x_2$$

$$X=\begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -2 & 4 \\ 0 & -12 & 12 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$R_3 \to 6R_3 - R_2$

$$\left[\begin{array}{ccc|c} -6 & 2 & 4 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-12x_2 = -12x_3$

$x_2 = x_3$

$-6x_1 - 2x_2 + 4x_3 = 0$

$-6x_1 + 2x_2 = 0$

$-6x_1 = -2x_2$

$x_2 = 3x_1$

$$\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 0 & 12 \\ -2 & 4-\lambda & 4 \\ -2 & 0 & 12-\lambda \end{vmatrix}$$

$= (-2-\lambda)\left[(4-\lambda)\cdot(12-\lambda)\right] + 12\left[2(4-\lambda)\right] = 0$

$4-\lambda\left[(-2-\lambda)(12-\lambda) + 24\right] = 0$

$4-\lambda = 0 \quad \lambda = 4$

$-24 + 2\lambda - 12\lambda + \lambda^2 + 24 = 0$

$\lambda^2 - 10\lambda = 0$

$\lambda(\lambda - 10) = 0$

$\lambda = 0, \; \lambda = 10$

$$\begin{bmatrix} -6 & 0 & 12 \\ -2 & 0 & 4 \\ -2 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ -2 & 0 & 4 & 0 \\ -2 & 0 & 8 & 0 \end{array}\right] \quad R_2 \rightarrow R_2 - R_3$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ 0 & 0 & -4 & 0 \\ -2 & 0 & 8 & 0 \end{array}\right] \quad R_3 \rightarrow 3R_3 - R_1$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 12 & 0 \end{array}\right] \quad R_3 \rightarrow 3R_3 + R_4$$

$$\left[\begin{array}{ccc|c} -6 & 0 & 12 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$-4x_3 = 0 \qquad x_3 = 0$

$-6x_1 + 12x_3 = 0$

$x_1 = 2x_3 = 0$

$x_2 = 1 \qquad X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

At $\lambda = 0$

$-2x_1 + 12x_3 = 0$

$4x_2 = 8x_3$

$x_2 = 2x_3$

$X = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

At $\lambda = 10$

$-12x_1 + 12x_3 = 0$

$-2x_1 - 6x_2 + 4x_3 = 0$

$-2x_1 + 2x_3 = 0$

$$\begin{bmatrix} -12 & 0 & 12 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At $\lambda = 4$, we get $\hat{\lambda}$ as

$$\begin{bmatrix} 0 & -2 & 4 \\ 0 & -6 & 12 \\ 0 & -2 & 8 \end{bmatrix}$$

$$-2x_2 + 4x_3$$

$$-6x_2 + 12x_3 = 0$$

$$-2x_2 + 8x_3 = 0$$

$$x_2 = 4x_3$$

$$-6x + 4x_3 + 12x_3 = 0$$

$$-12x_3 = 0$$

$$x_2 = 0$$

Put $x_1 = 1$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let Y = eigen vector of $\hat{\lambda}$.

at $\lambda = 0$

$$Y = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

$$X = PY$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

At $\lambda = 4$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

At $\lambda = 10$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Let A be an $n\times n$ matrix and let eigen vector of A be $\begin{bmatrix}1\\1\end{bmatrix}$, then corresponding orthonormal eigen vector is $\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

eg: $\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}4/\sqrt{41}\\5/\sqrt{41}\end{bmatrix}$

Quadratic form

The quadratic form Q in the components $x_1, x_2 \dots x_n$ for a vector x is defined as $Q = X^TAX$ where A is the coefficient matrix.

for eg: Let $X = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ $A = \begin{bmatrix}3 & 5\\5 & 2\end{bmatrix}$

$$Q = X^TAX.$$

Then corresponding quadratic form

$$Q = \begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}3 & 5\\5 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\begin{bmatrix}3x_1+5x_2 & 5x_1+2x_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$= 3x_1^2+5x_1x_2+5x_1x_2+2x_2^2$$

$$Q = 3x_1^2+10x_1x_2+2x_2^2$$

Consider the Quadratic form $Q = 3x_1^2+10x_1x_2+2x_2^2$

$$A = \begin{bmatrix}3 & 5\\5 & 2\end{bmatrix}$$

Let X be the matrix obtained by writing the orthogonal eigen vectors of A as column vectors.

$$X^Tx = Y$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \quad \text{--- (1)}$$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are eigen values of A.

$y_1, y_2 \dots y_n$ are components of y.

Here (1) is called principal axis form or cannonical form of the quadratic equation.

Qn: What kind of a conic section or pair of straight line is given by quadratic form:

Transform it to principal axis ~~from~~ form and express $X^T = [x_1 \; x_2]$ in terms of new vector $Y^T = [y_1, y_2]$

1) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

$$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$$

$X^T x = y$

Note: In quadratic form, the orthonormal eigen vectors are orthogonal.

ie $X^T = X^{-1}$

$$Q = \lambda_1 y_1^2 + \lambda_2 y^2$$

$$[A - \lambda I] = \begin{bmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{bmatrix} = 0$$

$$(7-\lambda)^2 - 9 = 0$$

$$40 + \lambda^2 - 14\lambda = 0$$

$\lambda = 4, 10$

$\therefore$ principal axis form is,

$$200 = 4y_1^2 + 10y_2^2$$

$$1 = \frac{y_1^2}{50} + \frac{y_2^2}{20} \longrightarrow \text{It represents an ellipse}$$

when $\lambda = 4$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$x_1 + x_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

when $\lambda = 10$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-x_1 + x_2 = 0$

$x_1 = x_2$

$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}} y_1 + 1/\sqrt{2}\, y_2$$

$$x_2 = 1/\sqrt{2}\, y_1 - 1/\sqrt{2}\, y_2$$

2) $x_1^2 - 12x_1x_2 + x_2^2 = 70$

$$A = \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix}$$

$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$$\begin{bmatrix} 1-\lambda & -6 \\ -6 & 1-\lambda \end{bmatrix} = 0$$

$(1-\lambda)^2 - 36 = 0$

$35 + \lambda^2 - 2\lambda = 0 \qquad \lambda = 7, -5$

$$70 = 7y_1^2 - 5y_2^2$$

$$\frac{y_1^2}{10} - \frac{y_2^2}{14} = 1$$

Gam: a hyperbola.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\lambda = 7$

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$x_1 + x_2 = 0$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = X_1 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda = -5$

$$\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 - x_2 = 0$

$-x_1 + x_2 = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 0$

$x_1 = x_2$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$X = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$x_1 = 1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2 \Rightarrow x_2 = -1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2$

$$\begin{bmatrix} 3 & 11 \\ 11 & 3 \end{bmatrix} = A$$

$$Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$\begin{bmatrix} 3-\lambda & 11 \\ 11 & 3-\lambda \end{bmatrix}$$

$$(3-\lambda)^2 - 11^2 = 0$$

$$(3-\lambda) = \pm 11$$

$$\lambda = -8, 14$$

when $\lambda = -8$

$$\begin{bmatrix} 11 & 11 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 + x_2 = 0$$

$$x_1 + x_2 = 0$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Equation for pair of straight lines

$$[ax^2 + 2hxy + by^2 = 0]$$

$$-8y_1^2 + 14y_2^2 = 0$$

$\lambda = 14$

$$\begin{bmatrix} -11 & 11 \\ 11 & -11 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$

$$x_2 = -1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2$$

If an eigen value λ is repeating k times then we choose corresponding k eigen vectors linearly independent. Since the system is homogeneous, then the rank of row reduced matrix is r and if there are n variables then no: of linearly independent solution is $(n-r)$ and remaining are zero vectors. No: k is called algebraic multiplicity of λ denoted by M_λ and $(n-r)$ is called geometric multiplicity denoted by m_λ. Then $\Delta_\lambda = M_\lambda - m_\lambda$ is called defect of λ.

1) find the value of (eigenvalue and eigenvector)

$$\begin{bmatrix} 6 & 5 & 2 \\ 2 & 0 & -8 \\ 5 & 4 & 0 \end{bmatrix}$$

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} (6-\lambda) & 5 & 2 \\ 2 & -\lambda & -8 \\ 5 & 4 & -\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(\lambda^2+32) - 5(-2\lambda+40) + 2(8+5\lambda) = 0$$

$$(6-\lambda)(\lambda^2+32) + 10\lambda - 200 + 16 + 10\lambda = 0$$

$$6\lambda^2 + 192 - 32\lambda - \lambda^3 + 20\lambda - 184.$$

$$-\lambda^3 + 6\lambda^2 - 12\lambda + 8 = 0$$

$$\lambda = 2, 2, 2.$$

$$\begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$4x_1 + 5x_2 + 2x_3 = 0$$

$$2x_1 - 2x_2 - 8x_3 = 0$$

$$5x_1 + 4x_2 - 2x_3 = 0$$

$$\left[\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 2 & -2 & -8 & 0 \\ 5 & 4 & -2 & 0 \end{array}\right]$$

$$\begin{bmatrix} 4 & 5 & 2 & 0 \\ 0 & -9 & -18 & 0 \\ 5 & 4 & -2 & 0 \end{bmatrix} \quad R_3 \to R_3 - 5/4 R_1$$

$$\begin{bmatrix} 4 & 5 & 2 & 0 \\ 0 & -9 & -18 & 0 \\ 0 & -9/4 & -18/4 & 0 \end{bmatrix}$$

$R_3 \to 4R_3 - R_2$

$$\left[\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 0 & -9 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$n = 3$
$r = 2$
$n - r = 1$

$-9x_2 - 18x_3 = 0$
$-9x_2 = 18x_3$
$x_2 = -2x_3$

$4x_1 - 10x_3 + 2x_3 = 0$
$4x_1 = 8x_3$
$x_1 = 2x_3$
$x_3 = 1$

$$X_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2) $$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Hint:
$\lambda = 5, -3, -3$
$n - r = 2$
Put $x_1 = 0$