$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

$$= \frac{1}{2} \times \frac{\pi}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

• S.T $\int_0^{\infty} \frac{dx}{1+x^6} = \pi/3$

$f(z) = \frac{1}{1+z^6}$

The poles are given by $1+z^6 = 0 \Rightarrow$

$z^6 = -1$

$z = (-1)^{1/6}$

$(-1)^{1/6} = 1^{1/6} \times e^{i\frac{(\pi+2k\pi)}{n}}$

$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right] = r^{1/n} e^{i\frac{(\pi+2k\pi)}{n}}$

$k = 0, 0, 1, 2, 3, 4, 5$

$(-1)^{1/6}$ = The pole are $z =$

$e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}, e^{i7\pi/6}, e^{i9\pi/6}, e^{i11\pi/6}$

The poles $\therefore z = e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}$ lies upper half plan

$z_1 = e^{i\pi/6}, \quad z_2 = e^{i3\pi/6}, \quad z_3 = e^{i5\pi/6}$

MODULE - 5

Rank of matrix

The rank of a $m \times n$ matrix A is non-negative integer 'r' such that -

1- There is atleast one $r \times r$ submatrix of A whose determinant is non-zero

2- The determinant of all square submatrix of A of order $\geq r+1$ is zero

eg:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$$

$|A| = 1[(-3 - -3)] - 2[-2+2] + 3[-6 - -6] = 0$

$r(A) < 3$

If det $= 0$

$r(A)$ lessthan max rank

Check the $r(A)$ less than 3, consider a 2x2 submatrix of A by eliminating a column and a row.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 3-4 = -1 \neq 0$$

$\therefore r(A) = 2$

o $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$|A| = 1[6-1] - 2[4-3] + 3[2-9]$

$= -18 \neq 0$

$\therefore R(A) = 3$

If $\neq 0$ $r(A)$ = order of matrix

o $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$

$|A| = 1[(4 \times 9) - (-36)] - 2[-18 - -18] - 3[12 - 12] = 0$

$\therefore$ 2x2 submatrix

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$

$\begin{vmatrix} 2 & 4 \\ -3 & -6 \end{vmatrix} = 0$

$\begin{vmatrix} 4 & 6 \\ -6 & -9 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 3 \\ -3 & -9 \end{vmatrix} = 0$

$\begin{vmatrix} 2 & 6 \\ -3 & -9 \end{vmatrix} = 0$

All 2x2 submatrix = 0 $r(A) < 2$

$\therefore$ $r(A) = 1$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}_{3\times 4}$$

Note
{ A $m \times n$
$r(A) \le \min(m,n)$ }

→ The rank of zero matrix is zero and the rank of other matrices is ≥ 1

3x3 submatrix -

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{vmatrix}$$

$= 1 \times (-2) - 1\,(1-6) + (-1)\,4$

$= -2 + 6 - 4 = \underline{\underline{0}}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} = 0$$ (If 2 columns or rows are equal = 0)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = 0$$

$\therefore$ $r(A) < 3$

Now let us check 2x2 submatrix

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \ne 0$$

$\therefore$ $r(A) = 2$

Row equivalent echelon matrix

A matrix A is said to be echelon form if the following conditions are hold;

1 - All zero rows, if any are at the bottom of matrix

2 - Each leading non-zero entry in a row is the right of the leading $\underbrace{\text{non-zero entry}}_{\text{(pivot)}}$ in the previous row

eg: (1) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 3 & 4 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

1- The rank of any matrix is the no of non-zero rows in any echelon matrix equivalent to A

2- Equivalent matrix have same rank

Elementary row operation (Transformation)

Elementary row operations in matrices are

(i) Inter change of any 2 rows

(ii) multiplication of any row by a non-zero constant

(iii) The addition of a constant multiple of the elements of any row to the corresponding elements of any other row

Q- find the rank of the following matrices using elementary row operations.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

Convert into echelon form.

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now the matrix is of echelon form. There are 3 non - zero terms (o) = 3 Hence the rank of matrix $r(A) = 3$

•
$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 + R_2 = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2 non-zero terms $r(A) = 2$

•
$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 9R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is a echelon form $r(A) = 3$

- $A = \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$
- $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

As per Syllabus

Linear system of equations

Gauss elimination method

A system of 'm' equations in (n) unknown of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

which can be written in matrix form $Ax = b$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$$

Augmented matrix,

$$\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the Augmented matrix and this matrix determines the system of eqn completely

In Gauss elimination method, we first reduce the Augmented matrix $\tilde{A}$ into the echelon form. Then we form an equivalent system from the echelon form of the augmented matrix. Then we solve this equivalent system by the back substitution method.

- Solve the following system of eqn by Gauss elimination method

$x_1 - x_2 + x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

$10x_2 + 25x_3 = 90$

$20x_1 + 10x_2 = 80$

Soln;

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$R_4 \to R_4 - 20R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

Inter changing R_2 & R_4 $\quad R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{3} R_2$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95/3 & 190/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$25 - \frac{-20}{3}$

$90 - \frac{1}{3} 80$

hence the equivalent system is –

$x_1 - x_2 + x_3 = 0$ — (1)

$30x_2 - 20x_3 = 80$ — (2)

$\frac{95}{3} x_3 = \frac{190}{3}$ — (3)

(3) $\Rightarrow x_3 = \frac{190}{95} = \underline{2}$

put $x_3 = 2$ in (2)

$30x_2 - 20 \times 2 = 80$

$30x_2 - 40 = 80$

$30x_2 = 80 + 40$

$x_2 = \frac{120}{30} = \underline{4}$

put $x_2 = 4$ in (1)

$x_1 - 4 + 2 = 0$

$x_1 - = \underline{0}$

$x_1 = \underline{2}$

$x_1 = 2, \quad x_2 = 4, \quad x_3 = \underline{2}$

System of linear eqns

- Inconsistant
  - No Solutions
- Consistant
  - Unique Solution
  - Infinite no. of Solutions

$$\tilde{A} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

$$\tilde{A} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(\tilde{A}) = 3$

$r(A) = 3$

$r(\tilde{A}) = r(A) =$ no. of unknowns {Unique solution}

$r(\tilde{A}) = r(A) <$ no. of unknowns {Infinite no. of Solutions}.

⇒ Existence and uniqueness of Solutions:

Fundamental theorem:

A linear system of eqns $Ax = b$ is consistant if and only if the coefficient matrix A and the augmented matrix $\tilde{A}$ have the same rank.

$$r(A) = r(\tilde{A})$$

If $r(A) \neq r(\tilde{A})$, the system is inconsistant or has no solution.

If $r(A) = r(\tilde{A}) = n$, number of unknowns the system has unique solution.

If $r(A) = r(\tilde{A}) < n$, the system has infinitely many solutions.

Theorem: (Homogenous system)

A homogenous system $Ax = 0$ is always consistant. i.e., $x = 0$ is always a solⁿ of $Ax = 0$. This solution is called trivial solution.

The homogenous system, $Ax = 0$ has a non-trivial solution, iff $r(A) < n$.

? Show that the system of eqns $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = 1$ is consistant and hence solve the system.

Ans.

$x + 2y - z = 3$

$3x - y + 2z = 1$

$2x - 2y + 3z = 2$

$x - y + z = 1$

$$\tilde{A} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

pivot elements.

If it is consistant, then $r(\tilde{A}) = r(A)$ where, $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

(R_2, R_3, R_4 1st term '0' [illegible])

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -2 \end{bmatrix}$$

pivot element

R_2, R_3, R_4 elements 0 - [illegible] pivot element [illegible]

Now, make the 2nd term in R_3, R_4 0 by using the pivot element -7.

$R_3 \to R_3 - \left(\frac{6}{7}\right) R_2$

$R_4 \to R_4 - \left(\frac{3}{7}\right) R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & 10/7 \end{bmatrix}$$

pivot

$2 - \frac{3}{7} 5$

$5 - \frac{6}{7} 5$

$-4 - \frac{6}{7} \cdot -8$

$-2 - \frac{3}{7} \cdot -8$

$R_4 \to R_4 + \frac{1}{5} R_3$ Multiplier

$$\text{Multiplier} = \frac{-1/7}{5/7} = -1/5$$

$-\frac{1}{7} + \frac{1}{5} \times \frac{5}{7}$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\frac{10}{7} + \frac{1}{5} \frac{20}{7}$

Rank of $\tilde{A} = r(\tilde{A}) = 4$ and $r(A) = 3$

Hence, the given system is inconsistent.

? Solve the following system of eqns:

$x_1 + x_2 - 2x_3 + 4x_4 = 5$

$2x_1 + 2x_2 - 3x_3 + x_4 = 3$

$3x_1 + 3x_2 - 4x_3 - 2x_4 = 1$

Ans. $\tilde{A} = \begin{bmatrix} 1 & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{bmatrix}$

$-3 - -4$

$-4 + 6$

$-2 - 12$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{bmatrix}$$

$-14 - -14$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$, $r(\tilde{A}) = 2$, $r(A) = 2$, therefore the system is consistent.

$r(\tilde{A}) = r(A) = 2 < 4$ no of unknown variable, hence, the system has infinite no. of solutions.

Hence the equivalent system is,

$$x_1 + x_2 - 2x_3 + 4x_4 = 5$$

$$x_3 - 7x_4 = -7$$

Here, x_1 and x_3 are pivot variables (starting variables). and x_2 and x_4 are free variables.

2 Pivot variables are 1 & 1.
free variables are the variables except 1 & 1

We find the solution by giving arbitrary values to the free variables.

put $x_2 = a$ and $x_4 = b$

$2 \Rightarrow x_3 - 7b = -7$

put $x_2 = a$ & $x_3 = -7+7b$, $x_4 = b$ in eqn(1),

$$x_1 + a - 2(-7+7b) + 4b = 5$$

$$x_1 + a + 14 - 14b + 4b = 5$$

$$x_1 + a - 10b = -9 \Rightarrow x_1 = -a + 10b - 9$$

Therefore, solution is of the form,

-1+10-1

$$x = \begin{bmatrix} -a+10b-9 \\ a \\ -7+7b \\ b \end{bmatrix}$$

put $a = 1$, $b = 1$, then

the solution is, $x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

{for correction, check the eqns by substituting all the values for x_1, x_2, x_3, x_4 ".}

Also, when $a = 0, b = 0$, $x = \begin{bmatrix} -9 \\ 0 \\ -7 \\ 0 \end{bmatrix}$, ie, we can get infinite no. of solutions.

? Show that the system of eqns,

$$x + y + z = 6$$

$$x + 2y + 3z = 14$$

$$x + 4y + 7z = 30$$

, are consistent & hence solve them.

Ans. $\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$ Augmented matrix

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - R_1$ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$

6 -

$R_3 \to R_3 - 3R_2$ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

non-full zero rows

$r(\tilde{A}) = 2$ and $r(A) = 2$

ie, $r(\tilde{A}) = r(A) = 2 < 3$ (unknown variables)

, ie the system has infinite no of solutions.

Therefore, the equivalent system is,

$x + y + z = 6$

$y + 2z = 8$

2 eqn-ൽ starting-ൽ variable z ആണ് free variable.

There are two pivot variables ~~x~~.

x & y are pivot variable & z is a free variable.

Put $z = a$, from (2)

$y + 2a = 8 \Rightarrow y = 8 - 2a$

Put $z = a$ & $y = 8 - 2a$ in (1)

$x + 8 - 2a + a = 6$

$x - a = -2$

$x = a - 2$

$\therefore$ the general solution is, $X = \begin{bmatrix} a-2 \\ 8-2a \\ a \end{bmatrix}$

particular solⁿ when $a = 0$ is;
when $a = 0$, $X = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$

? Solve the system of equations whose augmented matrix is given by,

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$$

Ans. $\tilde{A} = \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}$

$R_2 \to R_2 - \left(\frac{0.6}{3}\right) R_1$ ie, $R_2 \to R_2 - 0.2R_1$

$R_3 \to R_3 - \left(\frac{1.2}{3.0}\right) R_1$ $R_3 \to R_3 - 0.4R_1$

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{bmatrix}$$

$R_3 \to R_3 - \left(\frac{-1.1}{1.1}\right) R_2$

ie, $R_3 \to R_3 + R_2$ $\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore r(\tilde{A}) = r(A) = 2 < 4$, Unknown Variables

$\therefore$ the system is consistant.

$\therefore$ the system has infinite no. of solutions.

Therefore the Equivalent System is;

$$3x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \quad \cdots (1)$$

$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \quad \cdots\cdots (2)$$

x_1 and x_2 are the starting or pivot Variables.
And x_3, x_4 are the free Variables.

Put $x_3 = a$ —(a) $x_4 = b$ —(b)

then. (2) $\Rightarrow 1.1x_2 + 1.1a - 4.4b = 1.1$

$$1.1x_2 = 1.1 - 1.1a + 4.4b$$

$$\therefore x_2 = 1 - a + 4b \quad \cdots\cdots (3)$$

Substitute x_2, x_3, x_4 in (1) $\Rightarrow$

$$3x_1 + 2.0(1 - a + 4b) + 2.0a - 5.0b = 8.0$$

$$3x_1 = 8 + 5.0b - \cancel{2.0a} - 2.0 + \cancel{2.0a} - 2.0 \times 4b$$

$$3x_1 = 8 \quad 6.0 - 3b$$

$$\therefore x_1 = 2 - b$$

therefore the solution is;

$$X = \begin{bmatrix} 2 - b \\ 1 - a + 4b \\ a \\ b \end{bmatrix}$$

when $a = 0$, $b = 0$, then solution becomes;

$$X = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

when $a = 1$, $b = 1$, then the solution is;

$$X = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

? Solve the following system of eqns if it is consistant.

$$x + y + 2z = 2$$

$$2x - y + 3z = 2$$

$$5x - y + 8z = 10$$

Ans. $\tilde{A} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 8 & 10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 5R_1$ $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & -2 & 0 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

$\therefore r(\tilde{A}) = 3$, $r(A) = 2$

$r(\tilde{A}) \neq r(A)$, ie the system is inconsistent

it has no solution.

? For what values of λ and μ, the system of eqns

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$, have (i) no-solution

(ii) Unique solution

(iii) infinite no of solutions

Ans. $\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

final $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$

~~(i) Given system has no solution if $r(\tilde{A}) \neq r(A)$ ie $\lambda - 3 = 0$ $\mu - 10 \neq 10$ $\lambda = 3$ and $\mu \neq 10$~~

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$

$r(\tilde{A}) \neq r(A)$ Given system has no solution if $r(\tilde{A}) \neq r(A)$

ie $\lambda - 3 = 0$, $\mu - 10 \neq 10$

(ii) $r(\tilde{A}) = r(A) = 3$

The system has unique solution if,

$r(\tilde{A}) = r(A) = 3$

ie, $\lambda - 3 \neq 0$ and μ can take any value

ie, $\lambda \neq 3$ and μ can take any value.

(iii) the system has infinite solution.

if $\lambda = 3$ $\mu = 10$

? Investigate the values of λ and μ so that the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have

(i) Unique solution

(ii) Infinity solutions

(iii) No solutions.

Ans. $\tilde{A} = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$

$R_2 \to R_2 - (7/2)R_1$
$\cancel{R_3 \to R_3 - (2/2)R_1}$

$R_2 \to R_2 - \left(\frac{7}{2}\right)R_1$
$R_3 \to R_3 - R_1$ $\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15/2 & -39/2 & \frac{-47}{2} \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$

$3 - 7/2\ 3$

(i) The system has unique solution if,

$r(\tilde{A}) = r(A) = 3$

ie, $\lambda - 5 \neq 0$,

$\lambda \neq 5$, μ can take any value

(ii) the system has infinite solution, if

$r(\tilde{A}) = r(A) <$ no. of unknowns.

ie, $r(\tilde{A}) = r(A) < 3$

ie, $\lambda - 5 = 0$, $\mu - 9 = 0$

$\lambda = 5$, $\mu = 9$

(iii) No solution if,

$r(\tilde{A}) \neq r(A)$

ie, $r(\tilde{A}) = 3$ and $r(A) = 2$

$\lambda - 5 = 0$ and $\mu - 9 \neq 0$

$\lambda = 5$ and $\mu \neq 9$

Q. Solve the following system of eqns.

$x + y - 2z + 3w = 0$

$x - 2y + z - w = 0$

$4x + y - 5z + 8w = 0$

$5x - 7y + 2z - w = 0$

Ans. $A = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 5R_1$

$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & 16 \end{bmatrix}$ $R_3 \to R_3 - R_2$, $R_4 \to R_4 - 4R_2$ $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & +3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Here, , $r(A) = 2$.

$r(A) = 2 < 4$, unknown numbers.

∴ the given system has infinite no. of solutions.

Given system has non-trivial solution

$$x+y-2z+3w=0 \quad \text{——} (1)$$
$$-3y+3z-4w=0 \quad \text{——} (2)$$

x & y are pivot variables.

z & w are free variables.

Now, put $z=a$, $w=b$.

in (2) $\Rightarrow$ $-3y+3a-4b=0$

$$-3y = 4b-3a$$

$$\therefore y = -\tfrac{4}{3}b + a \quad \ldots\ldots (3)$$

(3) in (1) $\Rightarrow$ $x + -\tfrac{4}{3}b + a - 2a + 3b = 0$ $\quad -\tfrac{4}{3}+3$

$$x - a + \tfrac{5}{3}b = 0$$

$$\text{or } x = a - \tfrac{5}{3}b \quad \ldots\ldots (4)$$

ie, $X = \begin{bmatrix} a - \tfrac{5}{3}b \\ -\tfrac{4}{3}b + a \\ a \\ b \end{bmatrix}$ Hence, it is the solution

when $a=0$, $b=1$, $X = \begin{bmatrix} -\tfrac{5}{3} \\ -\tfrac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

when $a=1$, $b=0$, $X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

★ Matrix Eigen Value Problem:

→ Let A be a square matrix, λ be an unknown scalar and 'X' be an unknown column vector. In matrix Eigen Value problem, our aim is to find λ's and X's, satisfying the eqn $AX = \lambda X$

→ The λ's that satisfies the above eqn are called eigen values of X and the corresponding non-zero X's that satisfy the eqn are called Eigen vectors of matrix of A. Imp?

⇒ Characteristic Equation:

For any square matrix, A, the eqn $|A - \lambda I| = 0$ (characteristic eqn) of the matrix A, where λ is a parameter. Ans.

The roots of the characteristic eqn are called the eigen values of A.

The set of all eigen values is called

Spectrum of A.

The vector x satisfying the eqn $AX = \lambda X$ is called the eigen vector corresponding to the eigen value λ.

The eigen vectors are non-trivial solutions of the system of eqns $(A - \lambda I)X = 0$

The eigen vectors corresponding to one and the same eigen value λ of a matrix A together with 0 vector is called the eigen space of A.

Q) Find Eigen values and corresponding eigen vectors of the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Ans. The characteristic eqn is; $|A - \lambda I| = 0$

ie, $\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A - \lambda I$

ie, $\cancel{A - \lambda I} = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$

$\therefore |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$$(-5-\lambda)(-2-\lambda) - 4 = 0$$

$$+10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow \lambda = \frac{-7 \pm \sqrt{49 - 24}}{2}$$

$$= \frac{-7 \pm \sqrt{25}}{2} = -7 \pm 5/2$$

$\therefore \lambda = -1$ or $\lambda = -6$

$\therefore$ the eigen values are $\lambda_1 = -1$, $\lambda_2 = -6$.

$\boxed{\lambda^2 - \text{trace}(A)\,\lambda + |A| = 0}$ → characteristic eqn

→ eigen values and product gives determinant of A.
→ eigen values and sum gives the matrix and trace.
ie $\begin{bmatrix} -5 & \\ & -2 \end{bmatrix}$ $-5 + -2 = -7$

$\therefore$ characteristic eqn becomes,

$$\lambda^2 + 7\lambda + 6 = 0$$

→ To find eigen ~~matrices~~, vectors;

for $\lambda_1 = -1$, the eigen vectors are solution of the system $(A - \lambda_1 I)X = 0$

$$(A + I)X = 0$$

ie, $\left(\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $2\times2 \quad 2\times1$, 2×1

$\Rightarrow -4x_1 + 2x_2 = 0$ —(1)

$2x_1 - x_2 = 0$ —(2)

eqn (1) is a multiple of (2).

(1) $\Rightarrow -4x_1 + 2x_2 = 0 \Rightarrow 2x_2 = 4x_1$

$x_2 = 2x_1 \Rightarrow x_1 = x_2/2$

when $x_1 = a$, $x_2 = 2a$

Hence, eigen vector is of the form

ie, $X = \begin{bmatrix} a \\ 2a \end{bmatrix}$

when $a = 1$, $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

→ for $\lambda_2 = -6$, the eigen vector are solution of the system $(A - \lambda_2 I)X = 0$

Verification

$AX = \lambda X$

$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5+4 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

Hence, verified

ie, $(A + 6I)X = 0$

$\left(\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$ ....(1)

$2x_1 + 4x_2 = 0$ ....(2) eqn (1) is a multiple of (2).

$2x_1 = -4x_2 \Rightarrow x_1 = -2x_2$

when $x_2 = a \Rightarrow x_1 = -2a$

$\therefore X = \begin{bmatrix} -2a \\ a \end{bmatrix}$, when $a = 1$, $X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is one eigenvector

$AX = \lambda_2 X$

$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+2 \\ -4-2 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \end{bmatrix}$

$\lambda_2 X = -6\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \end{bmatrix}$

$\therefore AX = \lambda_2 X$. Hence verified.

Q. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Ans. The characteristic eqn is,

$|A - \lambda I| = 0$

$\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$

$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$(\lambda - 6)(\lambda - 1) = 0$

$\lambda = 1$ or $\lambda = 6$

∴ the Eigen values are $\lambda_1 = 1$, $\lambda_2 = 6$

→ To find Eigen vector for $\lambda_1 = 1$, the Eigen vectors are solution of the system,

$(A - \lambda_1 I)X = 0$

$(A - I)X = 0$

$\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ie, $4x_1 + 4x_2 = 0$ —(1)

$x_1 + x_2 = 0$ —(2), here (1) is a multiple of eqn (2). ie, $x_1 = -x_2 \Rightarrow$ when $x_1 = a$, $x_2 = -a$.

∴ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +a \\ -a \end{bmatrix}$, when $a = 1$, $X = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$ is one Eigen Vector

→ To find Eigen vectors for $\lambda_2 = 6$, the Eigen vectors are solution of the system,

$(A - \lambda_2 I)X = 0$

$(A - 6I)X = 0$

$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-x_1 + 4x_2 = 0$ ...(1)

$1x_1 - 4x_2 = 0$ ....(2)

There is only one equation.

$-x_1 + 4x_2 = 0 \Rightarrow x_1 = 4x_2$.

ie, when $x_2 = a$, $x_1 = 4a$

∴ $X = \begin{bmatrix} 4a \\ a \end{bmatrix}$, when $a = 1$, $X = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, is one Eigen vector.

? Find Eigen values and Eigen vectors of the system $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

Ans. The characteristic eqn is, $|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - 10 = 0$

$4 - \lambda - 4\lambda + \lambda^2 - 10 = 0$

$\lambda^2 - 5\lambda + 4$

? Find the Eigen Values and Eigen Vectors of $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

Ans: The characteristic eqn is, $|A - \lambda I| = 0$

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix} = 0 \Rightarrow$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 1-\lambda\left((2-\lambda)(3-\lambda)-1\right] + 1(3-\lambda-2)$$

$$= 1-\lambda\left[6-2\lambda-3\lambda+\lambda^2-1\right] + (1-\lambda)$$

$$= 1-\lambda\left(\lambda^2-5\lambda+5\right) + 1-\lambda$$

$$= \lambda^2-5\lambda+5-\lambda^3+5\lambda^2-5\lambda+1-\lambda$$

$$= -\lambda^3+6\lambda^2-11\lambda+6$$

$$= \lambda^3+6\lambda^2+11\lambda-6$$

$$0 = \lambda^3-6\lambda^2+11\lambda-6$$

when $\lambda = 1$, $1-6+11-6 = 12-12 = 0$

$\therefore (\lambda-1)$ is a factor.

$$(\lambda-1)(\lambda^2-5\lambda+6) = 0$$

$\lambda = 1$ or $\lambda = \dfrac{5 \pm \sqrt{25-24}}{2}$

$$\lambda = \frac{5 \pm 1}{2}$$

$\lambda = 1$ or $\lambda = 3, 2$

$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$, are Eigen Values.

$$\begin{array}{r|l} & \lambda^2-5\lambda+6 \\ \lambda-1 & \lambda^3-6\lambda^2+11\lambda-6 \\ & \lambda^3-\lambda^2 \\ \hline & -5\lambda^2+11\lambda-6 \\ & -5\lambda^2+5\lambda \\ \hline & 0+6\lambda-6 \\ & 6\lambda-6 \\ \hline & 0 \end{array}$$

To find Eigen Vector for $\lambda_1 = 1$, the Eigen Vectors are solution of the system,

$$(A - \lambda_1 I)X = 0.$$

$$(A - I)X = 0$$

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(1-\lambda)x_1 + x_2 + 2x_3 = 0$ — (1)

$-x_1 + (2-\lambda)x_2 + x_3 = 0$ — (2)

$x_2 + (3-\lambda)x_3 = 0$ — (3)

$0x_1 + x_2 + 2x_3 = 0$

$-x_1 + x_2 + x_3 = 0$

$0x_1 + x_2 + 2x_3 = 0$

There is only two equations.

From 1st two equations,

$0x_1+x_2+2x_3=0$

$-x_1+x_2+x_3=0$

By the method of cross multiplication,

Method of Cross Multiplication

$a_1x+a_2y+a_3z=0$

$b_1x+b_2y+b_3z=0$

$$\frac{x}{a_2b_3-a_3b_2}=\frac{-y}{a_1b_3-b_1a_3}=\frac{z}{a_1b_2-b_1a_2}$$

$$\frac{x_1}{1-2}=\frac{-x_2}{0--2}=\frac{x_3}{0--1}$$

$$\frac{x_1}{-1}=\frac{-x_2}{2}=\frac{x_3}{1}$$

$\Rightarrow x_1=-1,\ x_2=-2,\ x_3=1$ is one solution of the system.

Hence, the Eigen vector is $X_1=\begin{bmatrix}-1\\-2\\1\end{bmatrix}$

→ To find Eigen vector for $\lambda_2=2$, the Eigen vectors are solution of the system,

$(A-\lambda_2 I)X=0$

$(A-2I)X=0$

$$\begin{bmatrix}-1&1&2\\-1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-x_1+x_2+2x_3=0$

$-x_1+x_3=0$

$x_2+x_3=0$

Taking 1st two eqns, we get,

$-x_1+x_2+2x_3=0$

$-x_1+0x_2+x_3=0$

By the method of cross multiplication,

$$\frac{x_1}{1-0}=\frac{-x_2}{-1--2}=\frac{x_3}{0--1}$$

$$\frac{x_1}{1}=\frac{-x_2}{1}=\frac{x_3}{1}$$

ie, $x_1=1,\ x_2=-1,\ x_3=1$.

$\therefore X_2=\begin{bmatrix}1\\-1\\1\end{bmatrix}$

→ To find Eigen vector for $\lambda_3=3$, the Eigen vector is,

$(A-\lambda_3 I)X=0$

$$\begin{bmatrix}-2&1&2\\-1&-1&1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-2x_1 + x_2 + 2x_3 = 0 \quad \text{—— (1)}$

$-x_1 - x_2 + x_3 = 0 \quad \text{—— (2)}$

$0x_1 + x_2 + 0x_3 = 0 \quad \text{—— (3)}$

$x_2 = 0$ in (1) & (2) $\Rightarrow$

$-2x_1 + 2x_3 = 0$

$-x_1 + x_3 = 0$

$-\cancel{2}x_1 = -\cancel{2}x_3 \Rightarrow x_1 = x_3$

when $x_3 = a$, $x_1 = a$

ie, when $a = 1$, $x_1 = x_3 = 1$.

$\therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

★ Short cut Method :

Q. Find the Eigen Values and Eigen Vectors of

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. 5, -3, -3

Ans. The characteristic eqn is;

ie, $|A - \lambda I| = 0$

$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$= -2-\lambda\,[(1-\lambda)-\lambda - 12] - 2(-2\lambda - 6) - 3\,(-4 + 1\,(1-\lambda)]$

$= -2-\lambda\,(-\lambda + \lambda^2 - 12) + 4\lambda + 12 - 3\,(\overset{-3}{-4+1} - \lambda)$

$= \cancel{-2\lambda} + \cancel{2\lambda}$

$= 2\lambda - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 4\lambda + 12 \cancel{+ 12} + 9 + 3\lambda$

$0 = -\lambda^3 - \lambda^2 + 21\lambda + 45$

$= \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

~~when~~ $\lambda = -3$ is one root

$$\begin{array}{r|l} & \lambda^2 - 2\lambda - 15 \\ \lambda + 3 & \lambda^3 + \lambda^2 - 21\lambda - 45 \\ & \lambda^3 + 3\lambda^2 \\ \hline & -2\lambda^2 - 21\lambda \\ & -2\lambda^2 - 6\lambda \\ \hline & -15\lambda - 45 \\ & -15\lambda - 45 \\ \hline & 0 \end{array}$$

$(\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$

$(\lambda+3)(\lambda-5)(\lambda+3) = 0$

$\lambda = \underset{\lambda_1}{-3}, \underset{\lambda_2}{5}, \underset{\lambda_3}{-3}$

The Eigen Values are; $\lambda_1 = -3$, $\lambda_2 = 5$, $\lambda_3 = -3$

→ for $\lambda_1 = -3$:

The Eigen Vector X is given by

$(A - \lambda_1 I)X = 0$

$(A + 3I)X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ie, $x_1 + 2x_2 - 3x_3 = 0$
$2x_1 + 4x_2 - 6x_3 = 0$
$-x_1 - 2x_2 + 3x_3 = 0$

~~Taking 1st two eqns, we get,~~

$\cancel{\dfrac{x_1}{-12--12} = \dfrac{-x_2}{-6--6} = x_3}$

Here, there is only equation,

ie, $x_1 + 2x_2 - 3x_3 = 0$

put $x_2 = a$ and $x_3 = b$

ie, $x_1 + 2a - 3b = 0$

$x_1 = 3b - 2a$

ie, Solution is of the form, $X = \begin{bmatrix} 3b-2a \\ a \\ b \end{bmatrix}$

when, $a = 0$, $b = 1$

then $X = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \lambda_1$

when $a = 1$, $b = 0$

$X = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \lambda_3$ } for λ_1 & λ_3

when $a = 1$, $b = 1$

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

→ for $\lambda_2 = 5$

$(A - \lambda_2 I)X = 0 \Rightarrow (A - 5I)X = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$
$2x_1 - 4x_2 - 6x_3 = 0$
$-x_1 - 2x_2 - 5x_3 = 0$

Taking 1st two eqns, we get,

$$\frac{x_1}{-12-12} = \frac{-x_2}{42--6} = \frac{x_3}{28-4}$$

$$x_1/_{-24} = x_2/_{-48} = x_3/_{24}$$

$\therefore$ $x_1/_{-1} = x_2/_{-2} = x_3/_{1} \Rightarrow x_1 = -1$, $x_2 = -2$, $x_3 = 1$

$\therefore$ Solution of the form, (The Eigen Vector is;) $X = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

★ Short cut method to write characteristic eqn :

For 2x2 matrix A :

characteristic eqn is, $\lambda^2 - \text{trace}(A)\lambda + |A| = 0$

For 3x3 matrix A :

$\lambda^3 - \text{trace}(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$

where A_{ii} is the cofactor of a_{ii}.

⇒ Properties of Eigen Values :

(i) A square matrix, A and its transpose, A^T have the same eigen value.

(ii) The Eigen Values of a diagonal matrix or a triangular matrix are the same as its diagonal elements. eg: diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, eigen values are 1, 2, 5

Δ^{le} matrix: $\begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$, eigen Values are 1, 2, 5

(iii) The sum of Eigen Values of a matrix A is equal to its trace. ($\downarrow$ diagonal sum of A.)

(iv) The product of Eigen values of a matrix A is equal to its determinant.

(v) If λ is an Eigen Value of matrix A, then λ^m is an Eigen value of A^m.

If $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ are Eigen Values of A

Then Eigen Values of A^2 are $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9$

(vi) If $\lambda \neq 0$, is an Eigen Value of A then $1/\lambda$ is an Eigen value of A^{-1}.

(vii) If λ is an Eigen Value of A, then $|A|/\lambda$ is an Eigen Value of adj. A.

★ SYMMETRIC, SKEW SYMMETRIC & ORTHOGONAL MATRICES :

A square matrix, A is said to be

(i) Symmetric if $A^T = A$

(ii) Skew-Symmetric, if $A^T = -A$

(iii) Orthogonal, if $AA^T = I$ or $A^T = \ldots$

{Diagonal elements of skew symmetric = 0 }

⇒ Properties :

→ For any square matrix A, the matrix $A+A^T$ is a symmetric matrix and $A-A^T$ is a skew-symmetric matrix.

→ Any square matrix 'A' can be written as the sum of a symmetric matrix and a skew symmetric matrix as ;

$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

→ The eigen values of a symmetric matrix are real.

→ The ~~real~~ eigen values of a skew symmetric matrix are pure imaginary or zero.

→ The determinant of an orthogonal matrix has the value ±1

→ The eigen values of an orthogonal matrix are real or complex conjugates in pairs and have absolute value 1.

? Find the eigen values of A^3 and A^{-1}, adj. A

if $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

Ans. Here, A is an upper triangular matrix.

Diagonals ... ∴ Upper triangular matrix

∴ Eigen values of A are its diagonal elements.

$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$.

Now, the eigen values of A^3 are ;

$\lambda_1 = 1^3 = 1$

$\lambda_2 = 2^3 = 8$

$\lambda_3 = 3^3 = 27$

Now, the eigen values of A^{-1} are reciprocal of the eigen values of A.

∴ $\lambda_1 = 1, \ \lambda_2 = 1/2, \ \lambda_3 = 1/3$.

~~Now find the eigen values of A^3, A^{-1} and adj A~~

Now, eigen values of adj. A ;

Here, $|A| = 1 \times 2 \times 3 = 6$

The eigen values of adj. A are $|A|/1 = 6$, $|A|/2 = 6/2 = 3$, $|A|/3 = 6/3 = 2$

SIMILAR MATRICES

If A & B are two square matrices of same power, then A is said to be similar to B if there exist a non-singular matrix 'P' such that,

$$B = PAP^{-1} \quad \text{or} \quad B = Q^{-1}AQ$$

PROPERTIES:

Imp → Similar matrices have the same Eigen Values

→ If X is an eigen vector of A, then $Y = P^{-1}X$ is an Eigen vector of B, corresponding to the same Eigen Value.

✽ Diagonalisation of a Matrix:

The process of finding a similar diagonal matrix corresponding to a square matrix 'A' is called diagonalisation.

If an $n \times n$ square matrix A has 'n' linearly independent Eigen Vectors, then $D = X^{-1}AX$ is a diagonal matrix with Eigen Values of A as diagonal Elements. Here X is the matrix with these Eigen Vectors as column vectors.

Also, $D^m = X^{-1}A^mX$

where, $A^m = X D^m X^{-1}$

Q. Diagonalise the matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

Ans: The characteristic eqn is, $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$12 - 7\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 7\lambda + 10 = 0$

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$$(\lambda - 2)(\lambda - 5) = 0$$

The eigen values are $\lambda_1 = 2, \lambda_2 = 5$

The eigen Vector X_1 is given by,

$$(A - 2I)X_1 = 0.$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$

Now, $2x_1 = -x_2 \Rightarrow$ ~~when~~

$$\frac{x_1}{-1} = \frac{x_2}{2} \Rightarrow x_1 = -1, \quad x_2 = 2$$

∴ the Eigen Vector, $X_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

when $\lambda_2 = 5$, the eigen Vector X_2 is given by

$$(A - 5I)X_2 = 0$$

$$\begin{bmatrix}-1 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\left.\begin{aligned}-x_1+x_2&=0\\ 2x_1+-2x_2&=0\end{aligned}\right\}\text{ only one eqn}$$

$-x_1+x_2=0 \Rightarrow x_1=x_2$

$\therefore\ x_1/1 = x_2/1 \Rightarrow$ The eigen vector $X_2=\begin{bmatrix}1\\1\end{bmatrix}$

Therefore, the matrix X is;

$$X=\begin{bmatrix}-1 & 1\\ 2 & 1\end{bmatrix}$$

$|X| = -1-2 = -3$

Adj $X = \begin{bmatrix}1 & -1\\ -2 & -1\end{bmatrix}$

Adj A.
1st row elements interchange
2nd row -ve assign.

$\therefore X^{-1} = \frac{1}{-3}\begin{bmatrix}1 & -1\\ -2 & -1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-1 & 1\\ 2 & 1\end{bmatrix}$

adj A: Diagonal interchange
• other elements sign -ve

Now, $D = X^{-1}AX$

$$=\frac{1}{3}\begin{bmatrix}-1 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}-1 & 1\\ 2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}-1 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}-2 & 5\\ 4 & 5\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}6 & 0\\ 0 & 15\end{bmatrix}$$

$$D=\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}$$

Q. Diagonal the matrix $A = \begin{array}{c}\begin{matrix}a_{11} & a_{11} & a_{11}\end{matrix}\\ \begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\end{array}$

Ans The characteristic eqn is;

$$|A-\lambda I| = 0$$

$$\lambda^3 - \text{trace}(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$$

$$\lambda^3 - 6\lambda^2 + (3+3+3)\lambda - 4 = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$(\lambda-1)(\lambda-1)(\lambda-4) = 0$$

Cofactor of $a_{11} = A_{11}$

$A_{11} = \begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix} = 4-1 = 3$

Eigen values are $\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 4$

when $\lambda_1 = 1$, the eigen vector is given by

$$(A-I)X = 0$$

$$\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Only one eqn $x_1 - x_2 + x_3 = 0$

when $x_2 = 0,\ x_3 = 1$, we have $x_1 = -1$

$x_2 = 1,\ x_3 = 0$, we have $x_1 = 1$

$\therefore$ the eigen vectors are $X_1 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

$X_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - x_2 + x_3 = 0$$

$$-x_1 - 2x_2 - x_3 = 0$$

$$x_1 - x_2 - 2x_3 = 0$$

first two eqn's take ;

$$x_1/1 = x_2/-1 = x_3/1$$

$$X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$