MODULE-IV

RESIDUE INTEGRATION

★ **Zeros & Singularities of a function $f(z)$:**

The points at which a function $f(z)$ takes the value '0' is called zeroes of $f(z)$. In other words, a zero is a 'z' at which $f(z)=0$.

eg:- (i) $f(z)=(z-1)^2$

at $z=1 \Rightarrow f(z)=(1-1)^2=0$, $z=1$ is a zero of $f(z)=(z-1)^2$

(ii) $f(z)=z^2-1 \Rightarrow f(z)=1-1=0$ at $z=1$

$f(z)=(-1)^2-1=0$, at $z=-1$.

$\therefore z=1$ & $z=-1$ are the zeroes of $f(z)=z^2-1$

(iii) $f(z)=\sin z$.

$\sin z=0$ when $z=n\pi$, $n=0,\pm1,\pm2$ ....

There are infinite no. of zeroes.

(iv) $f(z)=e^z$

$f(z)=e^z$, has no finite zeroes.

★ A function $f(z)$ is singular or has a singularity at a point $z=z_0$, if $f(z)$ is not analytic at $z=z_0$, but every neighbourhood of $z=z_0$, contains points at which $f(z)$ is analytic.

We call $z=z_0$ an isolated singularity of $f(z)$, if $z=z_0$ has a neighbourhood without further singularities of $f(z)$.

Example;

(i) Consider $f(z)=\tan z$

$$f(z)=\sin z/\cos z$$

which is not analytic when $\cos z=0$

ie, when $z=(2n+1)\pi/2$, $n=0,\pm1,\pm2\ldots$

~~we can find a neighbourhood~~ ~~$\pi/2$~~

ie, the singularities of $f(z)=\tan z$, are $\pm\pi/2, \pm3\pi/2$

Moreover, $\tan z$ has isolated singularities at each of these points

$$\left(\frac{1}{z^3}\right)\frac{1}{(1-z)} = \frac{1}{z^3}\left[1+z+z^2+z^3\ldots\right]$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z^2 + \ldots$$

here the principle part contains only 3 terms hence $z=0$ is a pole of order 3.

⇒ Behaviour of an analytic functions Near poles & essential singularities

Theorem - 1 (Behaviour Near a pole)

If $f(z)$ is analytic and has a pole at $z=z_0$ then $|f(z)| \to \infty$ as $z \to z_0$ in manner

Theorem - 2 (Picard's Theorem)

If $f(z)$ is analytic and has isolated essential singularity at $z=z_0$, Then $f(z)$ takes on every value with almost one exceptional value in an arbitrary small ...

analytic that by assigning a suitable value $f(z_0)$
$f(z) \frac{\sin z}{z}$ become analytic at $z=0$ by defining $f(0)=1$

Theorem

The zeros of an analytic function $f(z)$ are isolated ie, each of them has a neighbourhood that contains no further zeros of $f(z)$

Theorem

Let $f(z)$ be analytic at $z=z_0$ and have a zero of nth order at $z=z_0$ Then $1/f(z)$ has a pole of nth order at $z=z_0$; and so does $\frac{h(z)}{f(z)}$ provided $h(z)$ is analytic at $z=z_0$ and $h(z_0)\neq 0$

Zero of order 'n'

An analytic function $f(z)$ has a zero of order (n) at $z=z_0$ if not only f but also the derivatives $f', f'', f''' \dots f^{(n-1)}$ are all zero at $z=z_0$ but $f^{n}(z_0)\neq 0$.

Example;

$f(z)=(z-1)^2$, $z_0=1$

$f(1)=0$

$f'(z)=2(z-1)$

$f'(10)=2(0-1)=-2$

$f''(z)=2$

$f''(10)=2\neq 0$

$z=1$ is zero of order 2

Example;

$f(z)=z^2+1$

$f(i)=-1+1=0$

$f'(z)=2z$

$f'(i)=2i\neq 0$

$z=i$ is a zero of order of 1

Example;

Example;

$$f(z) = (z-a)^3$$

$z=a$ is a zero of order 3.

$$f(a) = (a-a)^3 = 0$$

$$f'(z) = 3(z-a)^2 \times 1$$

$$f'(a) = 3(a-a)^2 = 0$$

$$f''(z) = 6(a-a) = 0$$

$$f'''(z) = 6$$

$$f'''(a) = 6 \neq 0$$

$z = a$ is zero of order 3

* A zero of order 1 is called simple zero

Example;

$$f(z) = \sin z$$

zeroes are ;

$0, \pm\pi, \pm 2\pi \ldots$

$f(z) = \sin(z)$, $f(0) = 0$

$f'(z) = \cos z$, $f'(0) = 1$

$z = 0$, of order 1

Hence $z = 0$ is a zero of order 1 (simple)

★ Residue Integration Method:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\cdots +$$

$$\underbrace{b_1/_{z-z_0}}_{\text{Residue}} + b_2/_{(z-z_0)^2} + \cdots\cdots$$

where, $a_n = \frac{1}{2\pi i} \oint_c (z^* - z_0)^{n+1} f(z^*) \, dz^*$

$b_n = \frac{1}{2\pi i} \oint f(z^*) (z^* - z_0)^{n+1} \, dz^*$

$b_1 = \frac{1}{2\pi i} \oint_c f(z^*) (z - z_0)^{1-1} \, dz^*$

$b_1 = \frac{1}{2\pi i} \oint_c f(z^*) \, dz^*$

$$\oint f(z)\, dz = 2\pi i b_1$$

→ The coefficient b_1 of $1/_{z-z_0}$ in the laurents series expansion of $f(z)$ is called the residue of $f(z)$ at $z = z_0$. And is denoted by, $b_1 = \underset{z=z_0}{\text{Res}} f(z)$

→ To evaluate $\oint_c f(z)\, dz$, where, c is a simple closed path.

→ Case- 1 :

If $f(z)$ is analytic inside and on c

By cauchy's integral theorem,

$$\oint_C f(z)\,dz = 0$$

Case-2:

If z_0 is a singularity of $f(z)$ lying inside C, then $\oint_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_0} f(z)$

Example;

Integrate the function $f(z) = z^{-4}\sin z$ counter clockwise around the unit circle $C: |z| = 1$.

Ans. $\oint_C f(z)\,dz = \oint_C \frac{\sin z}{z^4}\,dz$

The function $f(z) = z^{-4}\sin z$, is not analytic at $z=0$, which lies inside the unit circle, C.

To find residue;

$$z^{-4}\sin z = \frac{1}{z^4}\sin z$$

$$= \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^3} - \frac{1}{z \cdot 3!} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

$$z^{-4}\sin z = \frac{1}{z^3} - \frac{1}{3!\cdot z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

Hence, $\operatorname{Res}_{z=0} f(z) = -\frac{1}{3!} = \underline{\underline{-\frac{1}{6}}}$

$$\therefore \oint \frac{\sin z}{z^4}\,dz = 2\pi i \operatorname{Res}_{z=0} f(z)$$

$$= 2\pi i \times -\frac{1}{6}$$

$$= \underline{\underline{-\frac{\pi i}{3}}}$$

Q. Integrate $f(z) = \frac{1}{z^3 - z^4}$ clockwise around the circle $C: |z| = \frac{1}{2}$.

Ans. $f(z) = \frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)}$

$f(z)$ has singularities at $z=0$ and $z=1$. $z=0$ lies inside C and $z=1$ lies outside C. To evaluate this integral, we need to find the residue of $f(z)$ at $z=0$:

$$\frac{1}{z^3(1-z)} = \frac{1}{z^3}\left(1 + z + z^2 + \cdots\right)$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots$$

Hence $\operatorname{Res}_{z=0} f(z) - 1$

$$\therefore \oint_C \frac{1}{z^3-z^4}\,dz = -2\pi i \operatorname*{Res}_{z=0} f(z)$$

$$= -2\pi i \times 1$$

$$= -2\pi i$$

-ve indicates that the dirn is clockwise.

★ Formula to find Residue :

(i) Residue at a simple pole:

If $f(z)$ has simple pole $z = z_0$, then $\operatorname*{Res}_{z=z_0} f(z)$

$$= \lim_{z \to z_0} (z - z_0) f(z)$$

(ii) Residue at a pole of order m :

If $f(z)$ has a pole of order 'm' at $z = z_0$, then ~~residue~~ $\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}$$

~~(iii) Residue at a~~

In particular,

residue at a pole of order 2 is;

$$\frac{1}{(2-1)!} \lim_{z \to z_0} \left\{ \frac{d^{2-1}}{dz^{2-1}} \left[(z-z_0)^2 f(z) \right] \right\}$$

ie, $\lim_{z \to z_0} \frac{d}{dz} \left[(z-z_0)^2 f(z) \right]$

If z_0 is a pole of order 3;

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2!} \lim_{z \to z_0} \left\{ \frac{d^2}{dz^2} \left[(z-z_0)^3 f(z) \right] \right\}$$

? Find the residue of $f(z) = \frac{9z+i}{z^3+z}$ at $z = i$

Ans. $f(z) = \frac{9z+i}{z(z^2+1)} = \frac{9z+i}{z(z+i)(z-i)}$

$z = i$ is a simple pole of $f(z)$. Hence,

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} (z-i) f(z)$$

$$= \lim_{z \to i} \frac{(z-i)(9z+i)}{z(z+i)(z-i)}$$

Hence, $\operatorname*{Res}_{z=i} (f(z)) = \lim_{z \to i} (z-i) f(z)$

$$= \lim_{z \to i} (z-i) \frac{9z+i}{z(z+i)(z-i)}$$

$$= \lim_{z\to i} \frac{9z+i}{z(z+i)} = \frac{9i+i}{i(i+i)}$$

$$= 10i/2i^2 = \frac{10i}{-2} = -5i$$

? The function $f(z) = 50z/(z^3+2z^2-7z+4)$ has a pole of 2nd order at $z=1$. Find the residue at $z=1$.

Ans. $$\frac{50z}{z^3+2z^2-7z+4} = \frac{50z}{(z+4)(z-1)^2}$$

Residue of $f(z)$ at $z=1$ is,

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!} \lim_{z\to 1} \left\{ \frac{d}{dz} (z-1)^2 f(z) \right\}$$

$$= \lim_{z\to 1} \left\{ \frac{d}{dz} \cancel{(z-1)^2} \frac{50z}{\cancel{(z-1)^2}(z+4)} \right\}$$

$$= \lim_{z\to 1} 50 \left\{ \frac{d}{dz} \frac{z}{z+4} \right\}$$

$$= \lim_{z\to 1} 50 \left\{ \frac{(z+4)\times 1 - z\times 1}{(z+4)^2} \right\}$$

$$= \lim_{z\to 1} 50 \left\{ \frac{\cancel{z}+4-\cancel{z}}{(z+4)^2} \right\}$$

$$= 200 \lim_{z\to 1} \frac{1}{(z+4)^2}$$

$$= 200 \times 1/5^2 = 200/25 = 8$$

★ **Residue theorem:**
(Cauchy's residue theorem):

Let $f(z)$ be analytic inside a simple closed path 'c' and on c, except for finitely many singular points $z_1, z_2 \ldots\ldots z_k$. Then the $\int f(z)$ taken counter clockwise around 'c' is equal to $2\pi i$ times the sums of residues of $f(z)$ at $z_1, z_2, \ldots\ldots z_k$.

ie, $$\oint_C f(z)\,dz = 2\pi i \times \text{sum of residues at } z_1, z_2 \ldots\ldots z_k$$

$$= 2\pi i \times \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

⇒ **Second formula for residue at a simple pole:**

If $z = z_0$, is a simple pole of $f(z) = P(z)/q(z)$ where, $P(z_0) \neq 0$, then $\operatorname*{Res}_{z=z_0} f(z) = P(z_0)/q'(z_0)$

Examples;

(i) Evaluate $\oint_C \frac{z\,dz}{(z-4)(z-2)}$, where C is the circle

(a) $|z| = 1$ (b) $|z| = 3$.

Ans. $f(z) = \frac{z}{(z-4)(z-2)}$

∴ poles of $f(z) = 4, 2$.

ie, $f(z)$ has simple poles at $z = 2$ & $z = 4$.

(a) $|z| = 1$.

ie, both $z = 2$, $z = 4$ lies outside C.

∴ $\oint_C \frac{z\,dz}{(z-4)(z-2)} = 0$ (By cauchy's integral theorem.

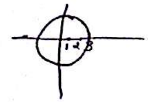
(b) $|z| = 3$. ie, $z = 2$ lies inside C. and $z = 4$

lies out side C $\int \frac{z}{(z-4)(z-2)}\,dz = 2\pi \operatorname{Res}_{z=2} f(z)$

$z = 2$ is a simple pole

$\operatorname{Res}_{z=2} f(z) = \lim_{z \to 2} (z-2) \times \frac{z}{(z-4)(z-2)}$

$= \frac{2}{2-4} = \frac{2}{-2} = -1$



$\therefore \oint f(z)\,dz = 2\pi i \times -1$

$= \underline{\underline{-2\pi i}}$

Q. Evaluate $\oint \frac{dz}{(z^2+4)^2}$ where, C is the circle $|z-i|=2$

Ans. $f(z) = \frac{1}{(z^2+4)^2}$

$(z^2+4)^2 = 0 \Rightarrow z = \pm 2i$

Here, $z=2i$ and $z=-2i$ are poles of order 2.

$|2i-i| = |i| = 1 < 2$.

$\therefore$, $z=2i$ lies inside $|z-i|=2$

Now, $|-2i-i| = |-3i| = 3 > 2$.

Therefore, $z=-2i$ lies outside $|z-i|=2$

Residue at $z=2i$;

$$\underset{z=2i}{\text{Res}} f(z) = \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 f(z)$$

$$= \lim_{z\to 2i} \frac{d}{dz}(z-2i)^2 \times \frac{1}{(z^2+4)^2}$$

$$= \lim_{z\to 2i} \frac{d}{dz}\cancel{(z-2i)^2} \times \frac{1}{\cancel{(z-2i)^2}(z+2i)^2}$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left(\frac{1}{(z+2i)^2}\right)$$

$$= \lim_{z\to 2i} \frac{-2}{(z+2i)^3}$$

$$\frac{d}{dx}\frac{1}{x^2} = \frac{d}{dx}x^{-2} = -2x^{-3} = \frac{-2}{x^3}$$

$$= \frac{-2}{(2i+2i)^3} = \frac{-2}{(4i)^3} = \frac{-2}{4^3 i^3}$$

$$= \frac{-2}{64\times -i} = \frac{1}{32i}$$

$$\therefore \oint_C \frac{dz}{(z^2+4)^2} = 2\pi i \times \operatorname{Res}_{z=2i} f(z)$$

$$= 2\pi i \times \frac{1}{32i} = \frac{\pi}{16}$$

? Evaluate $\oint_C \frac{e^z}{(z-1)^2(z-2)}$, where c is the circle $|z|=3$.

Here, numerator $\neq 0$, when $z = 2 \& 1$. $\therefore$ it is a pole of order 2 & 1.

~~Ans. $z=1$ is a pole of order 2 and $z=2$ is a simple~~ pole.

①

Problems :

1. Evaluate $\oint_{|z|=1} z^4 e^{1/z} dz$

2. Integrate $\tan z / z^2-1$ counterclockwise around the circle $|z|=3/2$.

3. Evaluate $\oint_C \frac{4-3z}{z^2-z} dz$, counterclockwise around any simple closed curve C such that

(a) 0 & 1 are inside C

(b) 0 inside, 1 outside

(c) 1 inside, 0 outside

(d) 0 and 1 are outside

Ans. (1) $f(z) = z^4 e^{1/z}$

$$= z^4\left[1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \cdots\cdots\right]$$

$$= z^4 + z^3/1! + z^2/2! + z/3! + 1/4! + 1/z \cdot 5! + 1/z^2 \cdot 6! \cdots\cdots$$

The principal part of the laurent's series contains infinite no. of terms. Hence, point z is 0 is an essential singularity. $z=0$ lies inside $|z|=1$.

$$\operatorname{Res}_{z=0} f(z) = \text{coefficient of } 1/z = 1/5! = 1/120.$$

$$\int_{|z|=1} z^4 e^{1/z} dz = 2\pi i \times \operatorname{Res}_{z=0} f(z)$$

$$= 2\pi i \times 1/120 = \pi i/60$$

②

2. $\tan z / z^2-1 = f(z)$

~~The integral~~ its singular points are ± 1.

$$f(z) = \tan z / z^2-1 = \tan z / (z-1)(z+1)$$

$f(z)$ has simple poles at $z=\pm 1$.

Here, $|z| = 3/2$ i.e, $z=\pm 1$ both lies inside $|z|=3/2$.

$$\operatorname{Res}_{z=1} f(z) = \operatorname{Res}_{z=1} f(z) \frac{\tan z}{z^2-1}$$

$$= \lim_{z\to 1} \frac{\cancel{(z-1)}\tan z}{\cancel{(z-1)}(z+1)}$$

$$= \frac{\tan 1}{1+1} = \frac{\tan 1}{2}$$

2nd formula can also be used here

ie $\frac{P(z_0)}{q'(z_0)} = \left[\frac{P(z)}{q'(z)}\right]_{z=z_0} = \left[\frac{\tan z}{2z}\right]_{z=1}$

Similarly,

$$\text{Res}_{z=-1} f(z) = \lim_{z\to -1} (z+1) \times \frac{\tan z}{(z-1)(z+1)}$$

$$= \frac{\tan(-1)}{-1-1} = \frac{-\tan 1}{-2} = \frac{\tan 1}{2}$$

Hence, ∴ $\displaystyle\int_{|z|=3/2} \frac{\tan z}{z^2-1}\,dz = 2\pi i \times \text{Sum of residues}$ (3)

$$= 2\pi i \times \left[\frac{\tan 1}{2} + \frac{\tan 1}{2}\right]$$

$$= 2\pi i \tan 1 = \underline{9.7855\, i}$$

(3) $f(z) = \frac{4-3z}{z^2-z} = \frac{4-3z}{z(z-1)}$

Singular points are $z=0$ & $z=1$.

ie, $f(z)$ has simple poles at $z=0$ & $z=1$.

$$\text{Res}_{z=0} f(z) = \lim_{z\to 0} \cancel{(z-0)} \frac{4-3z}{\cancel{z}(z-1)}$$

$$= \frac{4-0}{0-1} = -4$$

Also, $\text{Res}_{z=1} f(z) = \lim_{z\to 1} \cancel{(z-1)} \frac{4-3z}{z\cancel{(z-1)}}$

$$= \frac{4-3}{1} = \frac{1}{1} = 1$$

(a) case 1 : 0 & 1 are inside C.

$$\therefore \oint_C \frac{4-3z}{z(z-1)} = 2\pi i \left[\text{Res}_{z=0} f(z) + \text{Res}_{z=1} f(z)\right]$$

$$= 2\pi i(-4+1) = -6\pi i$$

(4)

(b) case-2 : 0 inside ; 1 outside C :

$$\oint \frac{4-3z}{z(z-1)} = 2\pi i\, \text{Res}_{z=0} f(z) = 2\pi i \times -4 = -8\pi i$$

(c) case-3 : 0 outside C, 1 inside.

$$\oint \frac{4-3z}{z(z-1)}\,dz = 2\pi i\, \text{Res}_{z=1} f(z) = 2\pi i$$

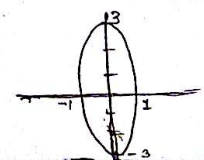
(d) $\displaystyle\oint \frac{4-3z}{z(z-1)}\,dz = 0$ {by cauchy's integral theorem}

? Evaluate $\displaystyle\oint_C \left(\frac{z e^{\pi z}}{z^4-16} + z e^{\pi/z}\right) dz$, where C is the ellipse $9x^2+y^2=9$.

Ans. $9x^2+y^2=9$

$x^2 + y^2/9 = 1$

or $x^2/1 + y^2/9 = 1$



$z^4 - 16 = 0 \Rightarrow (z^2)^2 - 4^2 = 0$

$(z^2+4)(z^2-4) = 0$

$\Rightarrow z = \pm 2, \pm 2i$ (4 roots).

Now, the terms of the integrant $\frac{z e^{\pi z}}{z^4-16}$ has simple poles at $z = \pm 2$ & $z = \pm 2i$.

2 & -2 lies outside the ellipse, and $2i$ & $-2i$ lies inside.

$\operatorname{Res}_{z=2i} f(z) =$

The poles $z = \pm 2$ lies outside the ellipse and $z = \pm 2i$ lies inside C $\operatorname{Res}_{z=2i} f(z) = \operatorname{Res}_{z=2i} \frac{z e^{\pi z}}{z^4-16}$

$= \operatorname{Res}_{z=2i} \frac{z e^{\pi z}}{(z+2)(z-2)(z+2i)(z-2i)}$

$= \lim_{z \to 2i} \cancel{(z-2i)} \frac{z e^{\pi z}}{(z+2)(z-2)(z+2i)\cancel{(z-2i)}}$

$= (2i)\, e^{2\pi i} \Big/ (2i+2)(2i-2)(2i+2i)$

$= \frac{2i}{-8 \times 4i} = \frac{-1}{16}$

$e^{i2\pi} = \cos 2\pi + i \sin 2\pi$
$= 1 + i \times 0$
$= 1$

? $\oint_C \left(\frac{z e^{\pi z}}{z^4-16} + z e^{\pi/z} \right) dz$

Ans. $\oint \frac{z e^{\pi z}}{z^4-16} dz + \oint_C z e^{\pi/z} dz$

$\Rightarrow \operatorname{Res}_{z=2i} f(z) = \operatorname{Res}_{z=-2i} \frac{z e^{\pi z}}{z^4-16}$

$= \operatorname{Res}_{z=-2i} \frac{z e^{\pi z}}{(z+2)(z-2)(z+2i)(z-2i)}$

$= \lim_{z \to -2i} \cancel{(z+2i)} \frac{z e^{\pi z}}{(z+2)(z-2)\cancel{(z+2i)}(z-2i)}$

$= \frac{(-2i)\, e^{i(-2\pi)}}{(-2i+2)(-2i-2)(-2i-2i)}$

$e^{i(-2\pi)} = \cos(-2\pi) + i \sin(-2\pi)$
$= 1 + i \times 0$
$= 1$

$= -2i \times 1 \Big/ ((-2i)^2 - 2^2) \times -4i$

$= \frac{-2i}{-8 \times -4i} = \frac{-1}{16}$

? $\oint_C \left(\frac{z e^{\pi z}}{z^4-16} + z e^{\pi/z} \right) dz$

Ans. $\oint_C \frac{z e^{\pi z}}{z^4-16} dz + \oint_C z e^{\pi/z} dz$.

The second integrant is $z e^{\pi/z}$.

$$ze^{\pi/z} = z\left[1+\frac{(\pi/z)}{1!}+\frac{(\pi/z)^2}{2!}+\frac{(\pi/z)^3}{3!}+\cdots\right]$$

$$= z\left[1+\frac{\pi}{1!\,z}+\frac{\pi^2}{2!\,z^2}+\frac{\pi^3}{3!\,z^3}+\cdots\right]$$

$$= z+\frac{\pi}{1!}+\frac{\pi^2}{2!\,z}+\frac{\pi^3}{3!\,z^2}+\cdots$$

Hence, $ze^{\pi/z}$ has an essential singularity at $z=0$. The residue at $z=0$ is the coefficient of $1/z$ in the laurents series expansion. i.e,

~~Residue~~

$$\operatorname*{Res}_{z=0} ze^{\pi/z} = \frac{\pi^2}{2!} = \frac{\pi^2}{2}$$

$$\therefore \oint_C\left(\frac{ze^{\pi/z}}{z^4-16}+ze^{\pi/z}\right)dz$$

$$= 2\pi i\,(\text{sum of residues})$$

$$= 2\pi i\left(-\frac{1}{16}+-\frac{1}{16}+\frac{\pi^2}{2}\right)$$

$$= 2\pi i\left(-\frac{2}{16}+\frac{\pi^2}{2}\right)$$

$$= 2\pi i\left(-\frac{1}{8}+\frac{\pi^2}{2}\right)$$

$$= 2\pi i\times\frac{1}{2}\left(-\frac{1}{4}+\pi^2\right)$$

$$= \pi i\left(\pi^2-\frac{1}{4}\right)$$

Appln: **Residue Integration of real integrants.**

Type-1: (Integrals of rational function of $\cos\theta$ and $\sin\theta$)

$$\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}$$

$$\cancel{\int_0^{2\pi} F(\cos\theta,\sin\theta)\,d\theta} \rightarrow \cancel{\oint_C f(z)\,dz}$$

→ To evaluate integrals of the type $\int_0^{2\pi} F(\cos\theta,\sin\theta)\,d\theta$ where $F(\cos\theta,\sin\theta)$ is a rational fn of $\cos\theta$ and $\sin\theta$ and is finite on the interval of integration.

If we set $z=e^{i\theta}$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(e^{i\theta}+\frac{1}{e^{i\theta}}\right) = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$z=e^{i\theta} \Rightarrow \frac{dz}{d\theta} = ie^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

Substituting these in the given problem, it becomes a problem in the residue integration method.

? Show that $\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = 2\pi$

Sol^n. Let C be the unit Circle, $|z|=1$

Then $z = e^{i\theta}$

put $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = \oint_{|z|=1} \frac{dz/iz}{\sqrt{2}-\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$= \oint_{|z|=1} \frac{dz}{iz\left(\sqrt{2}-\frac{z^2+1}{2z}\right)}$$

$$= \oint_{|z|=1} \frac{dz}{iz\frac{(2\sqrt{2}z-z^2-1)}{2z}}$$

$$= \frac{2}{i}\oint \frac{dz}{2\sqrt{2}z-z^2-1}$$

$$= \frac{2}{i}\oint \frac{dz}{-z^2+2\sqrt{2}z-1}$$

$$= -\frac{2}{i}\oint \frac{dz}{z^2-2\sqrt{2}z+1}$$

Take $f(z) = \frac{1}{z^2-2\sqrt{2}z+1}$

$z^2-2\sqrt{2}z+1=0$

$z = \frac{2\sqrt{2}\pm\sqrt{8-4}}{2} = \frac{2\sqrt{2}\pm 2}{2} = \sqrt{2}\pm 1$

ie, poles of $f(z)$ are $z=1+\sqrt{2}$, $z=1-\sqrt{2}$

$$\therefore \frac{1}{f(z)} = \frac{1}{[z-(\sqrt{2}-1)][z-(\sqrt{2}+1)]}$$

$$= \frac{1}{(z-\sqrt{2}+1)(z-\sqrt{2}-1)}$$

$f(z)$ has simple poles at $z=\sqrt{2}+1$ & $z=\sqrt{2}-1$

$z=\sqrt{2}+1$ lies outside the unit circle and $z=\sqrt{2}-1$ lies inside the unit circle.

$$\text{ie, } \operatorname{Res}_{z=\sqrt{2}-1} f(z) = \lim_{z\to\sqrt{2}-1} \cancel{[z-(\sqrt{2}-1)]}\frac{1}{\cancel{(z-\sqrt{2}+1)}(z-\sqrt{2}-1)}$$

$$= \lim_{z\to\sqrt{2}-1} \frac{1}{z-\sqrt{2}-1}$$

$$= \frac{1}{\sqrt{2}-1-\sqrt{2}-1} = \frac{1}{-2} = \frac{1}{-2}$$

$$\operatorname{Res}_{z=\sqrt{2}-1} f(z) = \frac{-1}{2}$$

$\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0)f(z)$

$$\text{Therefore, } -\frac{2}{i}\oint_C \frac{dz}{z^2-2\sqrt{2}z+1}$$

$$= -\frac{2}{i}\times 2\pi i \operatorname{Res}_{z=\sqrt{2}-1} f(z)$$

$$= \cancel{-\frac{2}{i}}\times 2\pi \cancel{i}\times\frac{-1}{\cancel{2}}$$

$$= 2\pi$$

Q Show that $\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = \frac{5\pi}{32}$

Ans. Let C be $|z|=1$

Then $z = e^{i\theta}$, $d\theta = dz/iz$

$\cos\theta = \frac{1}{2}(z + 1/z)$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = \oint_C \frac{dz/iz}{\left[5 - 3\times\frac{1}{2}(z+1/z)\right]^2}$$

$$= \oint \frac{dz/iz}{\frac{(3z-1)^2(z-3)^2}{4z^2}}$$

$$= \oint \frac{4z^2}{iz(3z-1)^2(z-3)^2}\, dz$$

$$= \frac{4}{i}\oint \frac{z\, dz}{(3z-1)^2(z-3)^2}$$

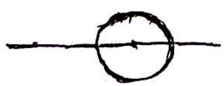
$$\left[5 - \frac{3}{2}(z+1/z)\right]^2 = \left[5 - \frac{3}{2}\left(\frac{z^2+1}{z}\right)\right]^2 = \left[5 - \frac{3(z^2+1)}{2z}\right]^2 = \left[\frac{10z - 3z^2 - 3}{2z}\right]^2 = \frac{(3z^2-10z+3)^2}{4z^2} = \frac{(3z-1)^2(z-3)^2}{4z^2}$$

Take $f(z) = \frac{z}{(3z-1)^2(z-3)^2}$

$f(z)$ has a poles of order 2 at $z = 1/3$ and $z = 3$.

$z = 1/3$ lies inside C &

$z = 3$ lies outside C.



$$\operatorname*{Res}_{z=1/3} f(z) = \lim_{z\to 1/3} \frac{d}{dz}\left\{(z-1/3)^2 \times \frac{z}{(3z-1)^2(z-3)^2}\right\}$$

$$= \lim_{z\to 1/3} \frac{d}{dz}\left\{\frac{(3z-1)^2}{3^2}\,\frac{z}{(3z-1)^2(z-3)^2}\right\}$$

$$= \lim_{z\to 1/3} \frac{1}{9}\,\frac{d}{dz}\left\{\frac{z}{(z-3)^2}\right\}$$

$$= \lim_{z\to 1/3} \times \frac{1}{9}\,\frac{(z-3)^2\times 1 - z\times 2(z-3)}{(z-3)^4}$$

$$= \frac{1}{9}\left[\frac{(1/3-3)^2 - 1/3\times 2(1/3-3)}{(1/3-3)^4}\right]$$

$$= \frac{1}{9}\left[\frac{(-8/3)^2 - 2/3\,(-8/3)}{(-8/3)^4}\right]$$

$$= \frac{\frac{1}{9}\left[\frac{64}{9} + \frac{16}{9}\right]}{\frac{4096}{81}} = \frac{1}{9}\left[\frac{80}{9}\right]\times\frac{81}{4096}$$

$$= 5/256.$$

$$\therefore \frac{4}{i}\oint \frac{z\,dz}{(3z-1)^2(z-3)^2} = \frac{4}{i}\times 2\pi i \times \operatorname*{Res}_{z=1/3} f(z)$$

$$= \frac{4}{i}\times 2\pi i \times \frac{5}{256} = \frac{40\pi}{256} = \frac{5\pi}{32}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = \frac{5\pi}{32}$$

? Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$

Ans. Let $C : |z| = 1$

Then $z = e^{i\theta}$, $d\theta = dz/iz$

$\cos\theta = \frac{1}{2}(z + 1/z) = \frac{1}{2}(e^{i\theta} + 1/e^{i\theta})$

$\cos 3\theta = \cancel{\frac{1}{2}(3z + 1/3z)}$

$= \frac{1}{2}(e^{i3\theta} + 1/e^{i3\theta})$

$= \frac{1}{2}((e^{i\theta})^3 + 1/(e^{i\theta})^3)$

$= \frac{1}{2}(z^3 + 1/z^3) = z^6 + 1/2z^3$

Substituting these in the given problem, we get,

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \int_{|z|=1} \frac{z^6+1/2z^3}{5-4\left(\frac{z^2+1}{2z}\right)} \, dz/iz$$

$$= \int_{|z|=1} \frac{z^6+1/\cancel{2}z^3}{\frac{10z-4z^2-4}{\cancel{2z}}} \, dz/iz$$

$$= \int_{|z|=1} \frac{z^6+1/z^2}{10z-4z^2-4} \, \frac{dz}{iz}$$

$$= \int_{|z|=1} \frac{z^6+1}{-iz^3(4z^2-10z+4)}\, dz = -\frac{1}{2i}\int_{|z|=1} \frac{z^6+1}{z^3(2z^2-5z+2)}\, dz$$

$$= -\frac{1}{2i}\int_{|z|=1} z^6+1/z^3(2z^2-1)(z-2)\, dz$$

$\cos n\theta = \frac{1}{2}(z^n + 1/z^n)$

Take $f(z) = z^6+1/z^3(2z-1)(z-2)$

$$\begin{array}{r|l} & 2z-1 \\ z-2 & 2z^2-5z+2 \\ & 2z^2-4z \\ \hline & -z+2 \\ & -z+2 \\ \hline & 0 \end{array}$$

The poles are $z=0$, $z=1/2$, & $z=2$

$z=0$ & $z=1/2$ lies inside $|z|=1$

$z=2$ lies outside $|z|=1$.

$$\operatorname{Res}_{z=1/2} f(z) = \lim_{z\to 1/2} (\cancel{z-1/2}) \times z^6+1/z^3\cancel{(2z-1)}(z-2)$$

$$= \frac{1}{2}\lim_{z\to 1/2} z^6+1/z^3(z-2)$$

$$= \frac{1}{2}\,(1/2)^6+1/(1/2)^3(1/2-2)$$

$$= -65/24$$

Now, $z=0$ is a pole of order 3.

$\therefore$ $\operatorname{Res}_{z=0} f(z) = 1/(3-1)! \lim_{z\to 0}\left[\frac{d^2}{dz^2} z^2 f(z)\right]$

$$= \frac{1}{2} \lim_{z\to 0} \left[\frac{d^2}{dz^2} z^8 \times \frac{z^6+1}{z^8(2z-1)(z-2)} \right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2} \frac{z^6+1}{(2z-1)(z-2)}$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2} \left(\frac{z^6+1}{2z^2-5z+2} \right)$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz} \left[\frac{(2z^2-5z+2)\,6z^5 - (z^6+1)(4z-5)}{(2z^2-5z+2)^2} \right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz} \left[\frac{8z^7 - 25z^6 + 12z^5 - 4z + 5}{(2z^2-5z+2)^2} \right]$$

$$12z^7 - 30z^6 + 12z^5 - 4z^7 + 5z^6 - 4z + 5$$
$$= 8z^7 - 25z^6 + 12z^5 - 4z + 5$$

$$\cancel{= \frac{1}{2} \lim_{z\to 0} (56z^6 - 150z^5 + 60z^4 - 4)}$$

$$\cancel{= \frac{1}{2} \times -4 = -2}$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4) - (8z^7-25z^6+12z^5-4z+5)\left(2(2z^2-5z+2)(4z-5)\right)}{(2z^2-5z+2)^4}$$

$$\cancel{112z^8 - 300z^7 + 120z^6 - 8z^2}$$

$$= \frac{2^2\cdot(-4) - (5)\cdot(4)(-5)}{2^4}$$

$$= \frac{-16 + 100}{2^4} = \frac{1}{2} \times \frac{84}{16}$$

$$= \frac{21}{8}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = -\frac{1}{2i} \times 2\pi i \,(\text{sum of residues})$$

$$= -\frac{1}{2i} \times 2\pi i \left[-\frac{65}{24} + \frac{21}{8} \right]$$

$$= -\pi \times -\frac{2}{24} = \frac{\pi}{12}$$

Type-II: [Integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$]

$$\int_{-\infty}^{\infty} f(x) = \lim_{b\to -\infty} \int_b^0 f(x)\,dx + \lim_{a\to\infty} \int_0^a f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

The integral $[-\infty, \infty]$ is equal to the $\lim_{R\to\infty} \int_{-R}^{R} f(x)dx$ and this limit is called principal value of $\int_{-\infty}^{\infty} f(x)dx$. If $f(z)$ is a rational function whose denominator has no real zeros and degree of denominator exceeds the degree of numerator by atleast 2. Then,

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \,[\text{sum of residues of } f(z) \text{ in the upper half}].$$

? Show that, $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$.

Ans. $f(x) = 1/(1+x^2)$, $f(z) = 1/(1+z^2) = \frac{1}{(z-i)(z+i)}$

The poles are $z = \pm i$

$z = i$ lies in the upper half plane and $z = -i$ lies the lower half plane.

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i} (z-i) f(z)$$

$$= \lim_{z\to i} \cancel{(z-i)} \frac{1}{\cancel{(z-i)}(z+i)}$$

$$= \frac{1}{i+i} = \frac{1}{2i}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \times \operatorname{Res}_{z=i} f(z) = 2\pi i \times \frac{1}{2i} = \pi.$$

? Show that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)} dx = \pi/5$

Ans. $f(z) = \frac{z^2}{(z^2+4)(z^2+9)}$ has simple poles at $z = \pm 2i$ and $z = \pm 3i$

$z = 3i$ and $z = 2i$ lies in the upper half plane

$$\operatorname{Res}_{z=2i} f(z) = \lim_{z\to 2i} (z-2i) f(z)$$

$$= \lim_{z\to 2i} \cancel{(z-2i)} \frac{z^2}{\cancel{(z-2i)}(z+2i)(z^2+9)}$$

$$= \frac{(2i)^2}{(2i+2i)((2i)^2+9)} = \frac{-4}{4i(-4+9)}$$

$$= \frac{-4}{4i \times 5} = \frac{-4}{20i} = \frac{-1}{5i}$$

$$\operatorname{Res}_{z=3i} f(z) = \lim_{z\to 3i} (z-3i) \times \frac{z^2}{(z^2+4)(z-3i)(z+3i)}$$

$$= \frac{(3i)^2}{((3i)^2+4)(3i+3i)}$$

$$= \cancel{} = \frac{-9}{(-9+4)6i} = \frac{-9}{-5(6i)}$$

$$= \frac{9}{30i} = \frac{3}{10i}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)} dx = 2\pi i \times \text{sum of residues of poles in upper half plane}$$

$$= 2\pi i \left[-\frac{1}{5!} + \frac{3}{10!} \right]$$

$$= 2\pi i \times \frac{1}{10 i} = \pi \times \frac{1}{5} = \pi/5$$

$\Rightarrow$ n^{th} root of a complex numbers:

$z = r(\cos\theta + i\sin\theta)$

$$z^{1/n} = r^{1/n} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right]$$

$k = 0, 1, 2 \ldots\ldots n-1$

$z = 1$

$z = 1 \cdot e^{i2\pi} = 1(\cos 2\pi + i \sin 2\pi)$.

$$1^{1/3} = 1^{1/3} \left[\cos\left(\frac{2\pi + 2k\pi}{3}\right) + i \sin\left(\frac{2\pi + 2k\pi}{3}\right) \right]$$

$$= \cos\left(\frac{2\pi(k+1)}{3}\right) + i \sin \frac{2\pi(k+1)}{3}, \quad k = 0$$

$k = 0, 1 \ldots\ldots$

$\cos 2\pi/3 + i \sin 2\pi/3$

$\cos 4\pi/3 + i \sin 4\pi/3$

$\cos 6\pi/3 + i \sin 6\pi/3 = \cos 2\pi + i \sin 2\pi = 1 + i \times 0 = 1$

? Show that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Ans. $\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

$f(z) = \frac{1}{1+z^4}$.

The poles of $f(z)$ are given by,

$1 + z^4 = 0 \Rightarrow z^4 = -1 \Rightarrow z = (-1)^{1/4}$

$\therefore$ 4^{th} roots of -1 are given by

$$(-1)^{1/4} = \cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right), \quad k = 0, 1, 2, 3$$

The poles of $f(z)$ is given by

$z^4 + 1 = 0 \Rightarrow z^4 = -1$

The roots are; $z = (-1)^{1/4}$

when $k = 0$

$(-1)^{1/4} = \cos \pi/4 + \sin \pi/4 = e^{i\pi/4}$

when $k = 1$

$(-1)^{1/4} = \cos 3\pi/4 + i \sin 3\pi/4 = e^{i3\pi/4}$

when $k = 2$

$(-1)^{1/4} = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}$

$= e^{i5\pi/4}$

$1 + z^4 = 0$

$z^4 = -1$

$z = (-1)^{1/4}$

no real roots so use complex root

$-1 = |1| e^{i\pi}$

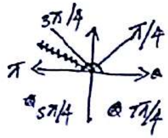
$-1 = 1(\cos\pi + i \sin\pi)$

$(\theta = \pi)$

$(-1)^{1/4} = \cos\frac{7\pi}{4} + i\sin 7\pi/4 = e^{i7\pi/4}$ //

here $z_1 = e^{i\pi/4}$ and $z_2 = e^{i3\pi/4}$

lies in the upper half plane



$\underset{z \to z_1}{Res} f(z) = \left[\frac{1}{d/dz\,(1+z^4)}\right]_{z=z_1}$

$= \left[\frac{1}{4z^3}\right]_{z=z_1}$

$= \frac{1}{4e^{(i\pi/4)3}}$

$= \frac{1}{4e^{i3\pi/4}}$

$= \frac{1}{4} e^{-i3\pi/4}$ //

$\underset{z \to z_2}{Res} f(z) = \left[\frac{1}{d/dz\,(1+z^4)}\right]_{z=z_2}$

$= \left[\frac{1}{4z^3}\right]_{z=z_2}$

$= \frac{1}{4(e^{i3\pi/4})^3} = \frac{1}{4\times e^{i9\pi/4}}$

$= 1/4 \times e^{-i9\pi/4}$ //

convert the point of $e^{-i3\pi/4}$ and $e^{-i9\pi/4}$ as same by using periodicity $\quad e^{-i3\pi/4} =$

$\frac{1}{2}\int \frac{dx}{1+x^4} = 2\pi i \times \left[\frac{1}{4}e^{-i3\pi/4} + 1/4\, e^{-i9\pi/4}\right]$ $\quad \cos 3\pi$ is $\sin 3\pi/4$

$\cos 5\pi - \pi/4$ is in ...

$= 2\pi i \times \left[-\frac{1}{4} e^{i\pi/4} + 1/4\, e^{-i\pi/4}\right] = [\cos\pi/4 + i\sin\pi/4]$

$\cos\pi/4 / i\sin\pi/4$

$= \frac{2\pi i \times 1}{\sqrt{2}}\left[-e^{i\pi/4} + e^{-i\pi/4}\right] = -e^{i\pi/4}$

$e^{-9\pi/4} = \cos 9\pi/4 - i\sin 9\pi/4$

taken out side

$= \frac{-\pi i}{2}\left[e^{i\pi/4} - e^{-i\pi/4}\right]$ $\quad \cos(2\pi + \pi/4 - \sin[2\pi + \pi/4$

$= \frac{-\pi i}{2}\left[e^{i\pi/4} - e^{-i\pi/4}\right]$ $\quad = \cos\pi/4 - i\sin\pi/4$

$= \frac{-\pi i}{2}\left[2i\sin\pi/4\right]$ $\quad = e^{-i\pi/4}$

$= \frac{-\pi}{2} \times 2i \times \frac{1}{\sqrt{2}}$ $\quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$= -\pi \times -1 \times \frac{1}{\sqrt{2}} = \pi/\sqrt{2}$ //

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2} \int \frac{dx}{1+x^4}$$

$$= \frac{1}{2} \times \frac{\pi}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

- S.T $\int_0^{\infty} \frac{dx}{1+x^6} = \pi/3$

$$f(z) \; \frac{1}{1+z^6}$$

The poles are given by $1+z^6 = 0 \Rightarrow$

$z^6 = -1$

$z = (-1)^{1/6}$

$(-1)^{1/6} = 1^{1/6} \times e^{i\left(\frac{\pi + 2k\pi}{n}\right)}$

$k = 0, 0, 1, 2, 3, 4, 5$

$(-1)^{1/6}$ = The pole are $z =$

$e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}, e$

$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$

$= r^{1/n} e^{i\left(\frac{\pi+2k\pi}{n}\right)}$

$e^{7\pi/6}, e^{i9\pi/6}, e^{i11\pi/6}$

The poles

$\therefore z = e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}$ lies upper half plane

$z_1 = e^{i\pi/6}, \; z_2 = e^{i3\pi/6}, \; z_3 = e^{i5\pi/6}$

MODULE - 5

Rank of matrix

The rank of a $m \times n$ matrix A is non-negative integer 'r' such that -

1- There is atleast one $r \times r$ submatrix of A whose determinant is non-zero

2- The determinate of all square submatrix of A of order $\geq r+1$ is zero

eg:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & -3 & -1 \end{bmatrix}$$

$$|A| = 1[(-3 - -3)] - 2[-2+2] + 3$$

$$[-6 - -6] = 0$$

If det = 0

$r(A) < 3$

$r(A)$ lessthan max rank