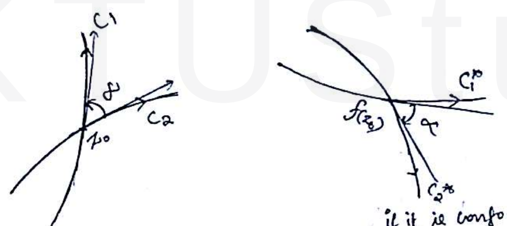
MODULE - II

CONFORMAL MAPPING

* A mapping $w = f(z)$ is said to be conformal at a point z_0 if it preserves the angle in both magnitude and sense (direction) b/w any two oriented curves C_1 and C_2 intersecting at z_0.

In otherwords, the images C_1^* and C_2^* of C_1 and C_2 makes the same angle as the curves themsel-ves in both magnitude and direction.



A mapping $w = f(z)$ is conformal at everypoint of its domain if it is conformal

* Theorem (Conformality of mapping by analytic functions):

The mapping $w = f(z)$ by an analytic function f is conformal except at critical points. ie, points at which $f'(z) = 0$.

For example;

(i) $f(z) = z^2$, is not conformal at the point $z = 0$,

Since $f'(z) = 2z$

$f'(z) = 0$ when $z = 0$.

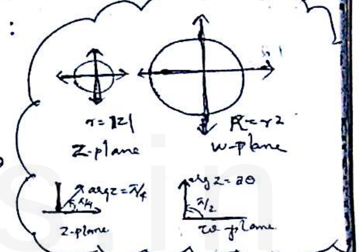
(ii) $f(z) = e^z$,

It is conformal at everypoint in the complex plane.

$f'(z) = e^z$, $e^z \neq 0$.

* SPECIAL MAPPINGS:

(i) The mapping $w = z^2$:



Let $z = re^{i\theta}$ and $w = Re^{i\phi}$

$w = z^2 \Rightarrow Re^{i\phi} = r^2 e^{i2\theta}$

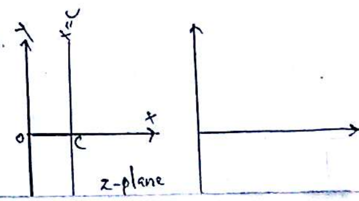
$\therefore R = r^2$ and $\phi = 2\theta$.

Under this mapping, the circles $|z| = r \Rightarrow r = r_0$ are mapped onto the circle $|w| = r^2$.

The rays $\theta = \theta_0$ are mapped onto rays $\phi = 2\theta$

* The image of the line $x = c$, where c is a constant. will be the parabola

$v^2 = 4c^2(c^2 - u)$



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* Proof: $w = z^2 = (x+iy)^2 = x^2-y^2+i2xy$

$u = x^2-y^2, \quad V = 2xy$.

(6)

when $x = c$, $u = c^2-y^2$ —(1) $V = 2cy$ —(2)

(2) $\Rightarrow V = 2cy \Rightarrow y = V/2c$ —(3)

Substitute (3) in (1) $\Rightarrow u = c^2 - V^2/4c^2$

$V^2/4c^2 = c^2-u$

$V^2 = 4c^2(c^2-u)$

* The image of horizontal line $y=c$, is the parabola $V^2 = 4c^2(c^2+u)$

Proof:

$w = z^2 = x^2-y^2+i2xy$

$u = x^2-y^2, \quad V = 2xy$

when $y = c$, $u = x^2-c^2$ —(1) $V = 2cx$ —(2)

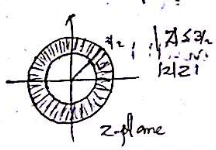
$x = V/2c$ —(3)

(3) in (1) $\Rightarrow u = V^2/4c^2 - c^2$

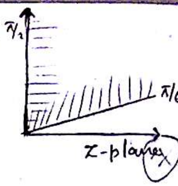
$V^2/4c^2 = u+c^2$

$V^2 = 4c^2(c^2+u)$. Hence, proved, which is a parabola.

* The region $1 \le |z| \le 3/2$ is mapped into $1 \le |w| \le 9/4$.

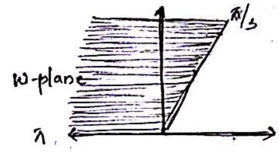


* $\pi/6 \le \theta \le \pi/2$ is mapped into

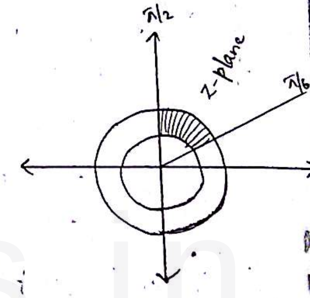


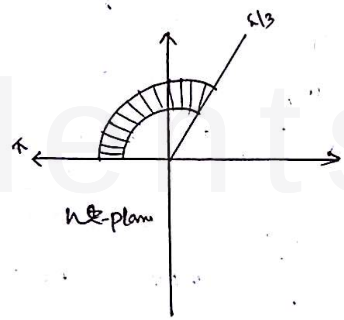
$\theta = \pi/6 \Rightarrow 2\times\pi/6 = \pi/3$

$\theta = \pi/2 \Rightarrow 2\times\pi/2 = \pi$



The region $\pi/3 \le \phi \le \pi$.





Q. Find the image of the Δ^{le} bounded by $x=1$, $y=1$, and $x+y=1$.

Ans we have $w = z^2$.

$w = (x+iy)^2 = x^2-y^2+2ixy$.

$u = x^2-y^2, \quad v = 2xy$

→ when $x = 1$,

$u = 1 - y^2$, $v = 2y$.

115 we have, when $x = c$, $v^2 = 4c^2(c^2 - u)$

∴ when $x = 1$, $v^2 = 4 \times 1^2 (1-u)$.

$v^2 = 4(1-u)$

Image of the line $x = c$ is $v^2 = 4c^2(c^2 - u)$

Similarly, image of the line $x = 1$ is, $v^2 = 4(1-u)$

→ Also, when $y = 1$

image, $v^2 = 4c^2(c^2 + u)$

$v^2 = 4 \times 1 (1^2 + u)$

$v^2 = 4(1+u)$

Similarly image of the line $y = 1$ is the parabola,
$v^2 = 4(1+u)$.

Now, to find image of $x + y = 1$

when $x + y = 1$, $u = x^2 - y^2 = (x+y)(x-y) = (x-y) \times 1$

$u = x - y$

$v = 2xy$.

Eliminating x and y from these two equations,

$(x+y)^2 = (x-y)^2 + 4xy$

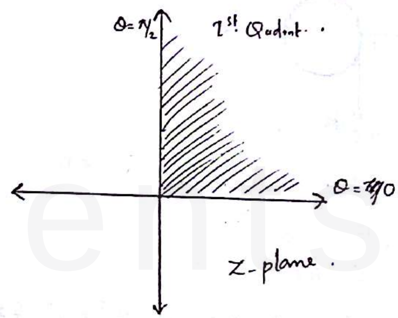
ie, $1^2 = u^2 + 2v$

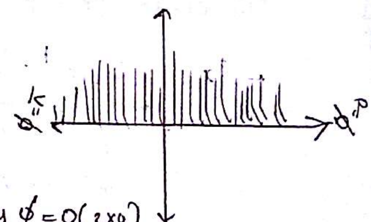
$u^2 + 2v = 1$ (parabola), which is a parabola

116 Therefore, the required image is the region bounded by the parabolas, $v^2 = 4(1-u)$, $v^2 = 4(1+u)$, $u^2 = 1 - 2v$.

Q Find the image of the first quadrant of z plane under the transformation $w = z^2$.

Ans.





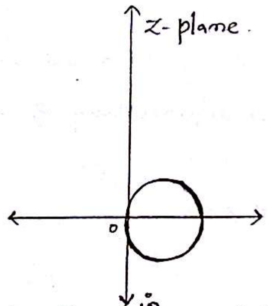
$0 \le \theta \le \pi/2$ is mapped into $0 \le \phi \le \pi$.

The ray $\theta = 0$ is mapped onto the ray $\phi = 0 (2 \times 0)$

And the ray $\theta = \pi/2$, is mapped on to the ray $\phi = \pi$ $(2 \times \pi/2)$

? Show that the transformation $w = z^2$ maps the circle $|z-1| = 1$ onto the Cardioid $\rho = 2(1+\cos\phi)$

Ans.



Let $z = re^{i\theta}$ and $w = \rho e^{i\phi}$

then, $w = z^2 \Rightarrow w = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$$\rho e^{i\phi} = r^2 e^{i2\theta}$$

$$\Rightarrow \rho = r^2, \quad \phi = 2\theta$$

The circle $|z-1| = 1$, is $x^2+y^2-2x = 0$

ie, $|x+iy-1| = |(x-1)+iy| = 1$

$$\Rightarrow \sqrt{(x-1)^2+y^2} = 1$$

$$x^2-2x+1+y^2 = 1$$

$$x^2+y^2-2x = 0$$

We know that, $x = r\cos\theta$, $y = r\sin\theta$

$z = re^{i\theta}$
$= r\cos\theta$

ie, $x^2+y^2-2x = 0$

$$(r\cos\theta)^2+(r\sin\theta)^2 - 2r\cos\theta = 0$$

$$r^2(\cos^2\theta+\sin^2\theta) - 2r\cos\theta = 0$$

$$\Rightarrow r^2 - 2r\cos\theta = 0 \Rightarrow r^{\cancel{2}} = 2\cancel{r}\cos\theta$$

from this $r = 2\cos\theta$

$$r^2 = 4\cos^2\theta$$

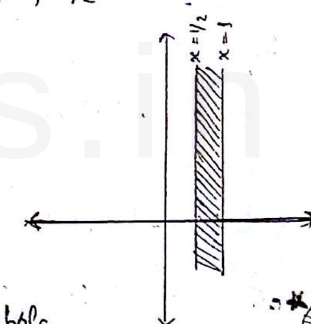
$$r^2 = 2(1+\cos 2\theta)$$

$\left\{ \cos^2\theta = \frac{1+\cos 2\theta}{2} \right.$

ie, $\rho = 2(1+\cos 2\theta)$

or $\rho = 2(1+\cos\phi)$ //

? find the image of the strip $\frac{1}{2} \le x \le 1$ under the mapping $w = z^2$



Ans Image of the strip when $x = \frac{1}{2}$ is, the parabola.

$$v^2 = 4c^2(c^2-u)$$

$$= 4 \times \frac{1}{4}\left(\frac{1}{4}-u\right)$$

$v^2 = \frac{1}{4} - u$ // is a parabola.

Similarly, when $x = 1$, Image is,

$$v^2 = 4c^2(c^2-u)$$

$$= 4 \times 1(1-u)$$

$v^2 = 4(1-u)$ is a parabola.

thus the infinite strip $1/2 \le x \le 1$ is mapped on to the region bounded by the parabolas $v^2 = \frac{1}{4} - u$ and $v^2 = 4(1-u)$

(vi) The mapping $w = e^z$:

$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$\therefore u = e^x \cos y, \quad v = e^x \sin y$

$|e^z| = \sqrt{(e^x\cos y)^2 + (e^x \sin y)^2}$

$= \sqrt{e^{x^2}(\cos^2 y + \sin^2 y)} = \sqrt{(e^x)^2} = e^x$

$\operatorname{Arg} w = \tan^{-1}(v/u) = \tan^{-1}\left(\frac{e^x \sin y}{e^x \cos y}\right)$

$= \tan^{-1}(\tan y)$

$\operatorname{Arg} w = y$

ie, $|e^z| = e^x$

$\arg(e^z) = y$

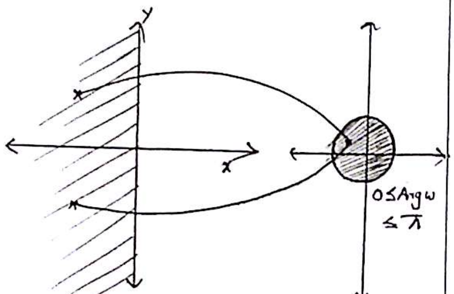
when, $x = c$, $|w| = e^x \Rightarrow |w| = e^c$.

The line $x = c$ where c is a constant is mapped on to the circle $|w| = e^x$

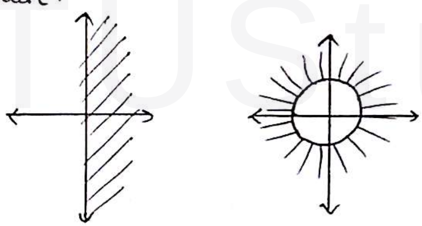
The horizontal line $y = c$, is mapped on to the ray $\arg w = c$

The image of the upper half plane is the upper half plane.

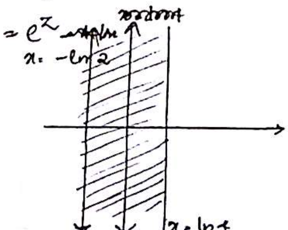
→ The image of the left half plane is the interior of the unit disc.



The image of the right half plane is the exterior of the unit disc.



Q) Find the image of the region $-\log 2 \le x \le \ln 4$ under the mapping $w = e^z$

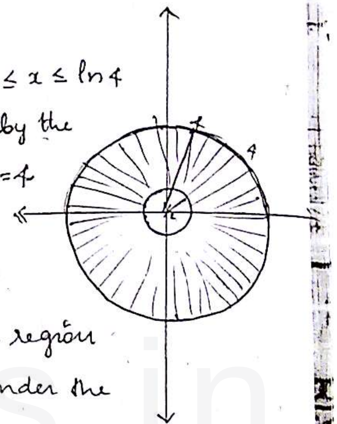


The image of the line $x = -\ln 2$ is the circle $|w| = e^{-\ln 2}$

$\Rightarrow |w| = e^{\ln 2^{-1}} \Rightarrow 2^{-1} = 1/2$

Similarly, the image of the line $x = \ln 4$ is the circle $|w| = e^{\ln 4} \Rightarrow |w| = 4$.

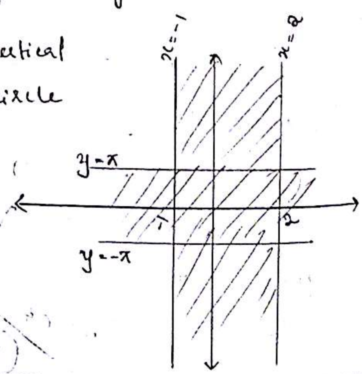
Hence, the region $-\ln 2 \le x \le \ln 4$ is the region bounded by the circles $|w| = 1/2$ and $|w| = 4$



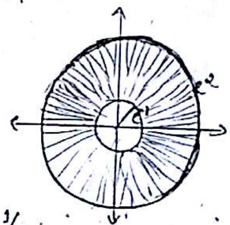
Q) Find the image of the region $-1 \le x \le 2$, $-\pi \le y \le \pi$ under the mapping $w = e^z$.

Ans $x = -1$, $x = 2$ $y = -\pi$, $y = \pi$

The image of the vertical line $x = -1$ is the circle $|w| = e^{-1}$ and the image of the vertical line $x = 2$ is the circle $|w| = e^2$.



$y=-\pi$ and $y=\pi$ are mapped on the rays $\arg w=-\pi$ and $\arg w=\pi$ respectively.



(iii) The mapping $w = w = 1/z$

$w = 1/z = 1/x+iy = x-iy/x^2+y^2$

$= x/x^2+y^2 - iy/x^2+y^2$

$\therefore u = x/x^2+y^2, \quad v = -y/x^2+y^2$

Therefore,

*A straight line of the form $x=c$ will be transformed into a circle $c(u^2+v^2)-u=0$.

Ans $\Rightarrow$ When $x=c$,

$u = c/c^2+y^2$ — ①, $v = -y/c^2+y^2$ — ②

$\Rightarrow c^2+y^2 = c/u$

$y^2 = c/u - c^2$ — (4)

$\cancel{(c^2+y^2)v = -y}$

$\cancel{y = -v(c^2+y^2)}$

$v^2 = y^2/(c^2+y^2)^2$ — ③

Put (4) in (3) $\Rightarrow$

$v^2 = c/u - c^2 / (c^2 + c/u - c^2)^2$

$v^2 = c/u - c^2 / (c/u)^2$

$= \left(\frac{c}{u} - c^2\right) \times \frac{u^2}{c^2} = c/u\, u^2/c^2 - c^2 u^2/c^2$

$v^2 = u/c - u^2$

$u^2+v^2 = u/c \Rightarrow c(u^2+v^2) = u$

$c(u^2+v^2) - u = 0$

H.W ? Show that under the mapping $w=1/z$, a straight line of the form $y=c$ will be transformed into a circle of the form $c(u^2+v^2)+v=0$.

* The half plane $x>c$ is mapped into the region $(u-1/2c)^2+v^2<(1/2c)^2$

* The image of the circle $x^2+y^2=c^2$ will be the circle $u^2+v^2=1/c^2$

$u = x/x^2+y^2$ & $v = -y/x^2+y^2$

$x = u/u^2+v^2$ & $y = -v/u^2+v^2$

$x^2+y^2 = c^2$

$\left(u/u^2+v^2\right)^2 + \left(-v/u^2+v^2\right)^2 = c^2$

$u^2+v^2 / (u^2+v^2)^2 = c^2$

? Find the image of the region $x>1$, $y>1$ under the mapping $|w| = 1/z$.

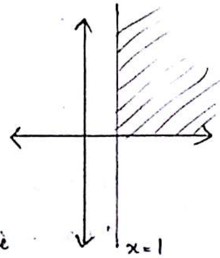
Ans. $x>1 \Rightarrow \frac{u}{u^2+v^2} > 1$

H14

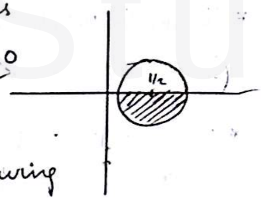
$u^2+v^2-u<0$

or $(u-\frac{1}{2})^2+v^2<(\frac{1}{2})^2$,

which is the interior of the circle, $(u-\frac{1}{2})^2+v^2 = \frac{1}{4}$



The image of $y>0$ is $v<0$, which is the lower half plane. Thus image of the strip (region) $x>1$, $y>0$ is the region bounded by $v<0$ and $(u-\frac{1}{2})^2+v^2 = \frac{1}{4}$



? Find the image of the following regions under the mapping $w = 1/z$.

(i) $\frac{1}{4} < y < \frac{1}{2}$

(ii) $0 < y < \frac{1}{2}$

(iii) $|z-2i| = 2$

(iv) $|z-3| = 5$

(19)

H.W. Ans:) when $y=c$,

$u = \frac{x}{c^2+x^2}$ —(1), $v = \frac{-c}{c^2+x^2}$ —(2)

H15

$c^2+x^2 = x/u$

$x^2 = x/u - c^2$ —(4)

$v^2 = \frac{c^2}{(c^2+x^2)^2}$ —(3)

Substitute (4) in (3) $\Rightarrow$, $v^2 = \frac{c^2}{(c^2 + x/u - c^2)^2}$

$v^2 = c^2/\frac{x^2}{u^2}$

$v^2 = \frac{u^2c^2}{x^2} = \frac{u^2c^2}{x/u - c^2}$

from (2) $\Rightarrow v(c^2+x^2) = -c$

or $c^2+x^2 = -c/v$, $x^2 = -c/v - c^2$ —(4)

We have (1), $u = \frac{x}{c^2+x^2}$ —(3)

(4) in (3) $\Rightarrow u = \frac{x}{c^2 + -c/v - \cancel{c^2}}$

$u = \frac{x}{-c/v}$

$u^2 = \frac{x^2}{c^2/v^2} = \frac{v^2x^2}{c^2}$

$u^2 = v^2x^2/c^2 \Rightarrow u^2 = \frac{v^2}{c^2}\times(-c/v - c^2)$

$= \frac{v^2}{c^2}\times -c/v - \frac{v^2}{c^2}c^2$

$u^2 = -v/c - v^2$

$u^2+v^2 = -v/c$

or $c(u^2+v^2)+v=0$ Hence proved.

(iii) $|z-2i|=2$

let $w=1/z = u+iv$

Also, we have $z=x+iy$

Now $|z-2i|=2$

$|x+iy-2i|=2$

$|x+i(y-2)|=2$

$\sqrt{x^2+(y-2)^2} = 2 \Rightarrow \sqrt{x^2+y^2-4y+4} = 2$

Squaring bothsides,

$x^2+y^2-4y+4=4$

$x^2+y^2-4y+4-4=0 \Rightarrow x^2+y^2-4y=0$

Substitute for x^2 and y^2,

$u^2/(u^2+v^2)^2 + v^2/(u^2+v^2)^2 + 4\, v/(u^2+v^2)=0$

$u^2+v^2/(u^2+v^2)^2 + 4v/(u^2+v^2) = 0 \Rightarrow u^2+v^2+4v(u^2+v^2)=0$

$(u^2+v^2)(1+4v)=0$

$1+4v=0$

which is the straight line.

$u^2+v^2+4v(u^2+v^2)=0$

$(u^2+v^2)(1+4v)=0$

$1+4v=0$, which is a straight line.

Thus the image of the circle $|z-2i|=2$ is the straight line $1+4v=0$.

(4) $|z-3|=5$

let $w=1/z = u+iv$

Then $x = u/(u^2+v^2)$ and $y = v/(u^2+v^2)$

And also $z=x+iy$

$\therefore |z-3|=5$

$|(x+iy)-3|=5 \Rightarrow |(x-3)+iy|=5$

$\sqrt{(x-3)^2+y^2} = 5$

Squaring bothsides,

$(x-3)^2+y^2=25$

$x^2-6x+9+y^2=25$

$x^2+y^2-6x-16=0$

$x^2+y^2-6x=16$

Substitute for x^2+y^2,

$u^2/(u^2+v^2)^2 + v^2/(u^2+v^2)^2 - 6\, u/(u^2+v^2)=16$

$$\frac{u^2+v^2}{(u^2+v^2)^2} - \frac{6u^2}{(u^2+v^2)} = 16$$

$$\frac{1}{u^2+v^2} - \frac{6u}{u^2+v^2}$$

$$\frac{u^2+v^2-6u(u^2+v^2)}{(u^2+v^2)^2} = 16$$

$$\frac{u^2+v^2(1-6u)}{(u^2+v^2)^2} = 16 \Rightarrow 1-6u = 16(u^2+v^2)$$

$$\frac{1-6u}{16(u^2+v^2)} = 0 \Rightarrow 1-6u=0$$

which is a straight line. Then the image of the circle $|z-3|=5$ is the straight line

1) $\frac{1}{4}<y<\frac{1}{2}$

Ans. $\frac{1}{4}<y$ and $y<\frac{1}{2}$

let $w = \frac{1}{z} = u+iv$

Then $x = \frac{u}{u^2+v^2}$ and $y = -\frac{v}{u^2+v^2}$

• $\frac{1}{4}<y \Rightarrow \frac{1}{4} < -\frac{v}{u^2+v^2}$

$u^2+v^2 < -4v$

$u^2+v^2+4v<0$, which is the interior part of a circle

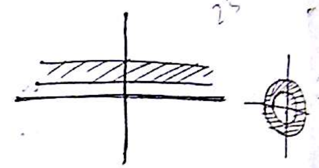
• $y<\frac{1}{2}$

$\Rightarrow -\frac{v}{u^2+v^2} < \frac{1}{2}$

$\Rightarrow -2v < u^2+v^2$

$u^2+v^2 > -2v$

$u^2+v^2+2v>0$



which is the exterior part of a circle. Thus the image of the region $\frac{1}{4}<y<\frac{1}{2}$ is mapped in the region between the circles $u^2+v^2+4v=0$ and $u^2+v^2+2v=0$

(2) $0<y<\frac{1}{2}$

let $w = \frac{1}{z} = u+iv$

Then $x = \frac{u}{u^2+v^2}$ and $y = -\frac{v}{u^2+v^2}$

• $0<y$

$0 < -\frac{v}{u^2+v^2}$

$v>0$ ∴ The image of the $0<y$ is $v>0$ which is the upper half plane.

• $y<\frac{1}{2}$

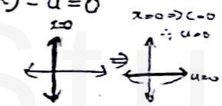
$\Rightarrow -\frac{v}{u^2+v^2} < \frac{1}{2} \Rightarrow -2v < u^2+v^2$

$u^2+v^2+2v>0$

The image of the $y < 1/2$ is $u^2 + v^2 - 2u > 0$ which is the exterior part of a circle and passing through the origin and centre at $(1,0)$. Thus the image of the region $0 < y < 1/2$, is the region mapped into the exterior part of a circle passing through the origin and centre at $(1,0)$ and above the real axis.

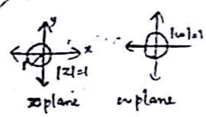
✱ Properties of the inversion mapping $w = 1/z$:

(i) A straight line of the form $x = c$ vertical line is mapped on to the circle $c(u^2+v^2) - u = 0$



(ii) A straight line of the form $y = c$ (horizontal) is mapped onto a circle of the form $c(u^2+v^2) + v = 0$

(iii) The mapping $w = 1/z$ maps the unit circle $|z| = 1$ onto the unit circle $|w| = 1$.



$w = 1/z = 1/re^{i\theta} = 1/r \, e^{-i\theta}$

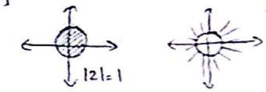
$w = \rho e^{i\phi}$

$\rho = 1/r \quad \phi = -\theta$

$|w| = 1/r$

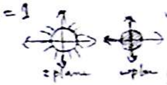
If $|z| = 2$, then in w plane $|w| = 1/2$.

(iv) The interior of the unit circle $|z| = 1$ is mapped onto the exterior of the unit circle $|w| = 1$

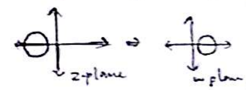


$r > 1$

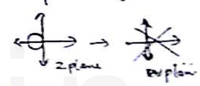
(v) The exterior of the unit circle $|z| = 1$ is mapped onto the interior of the unit circle $|w| = 1$



(vi) A circle not passing through origin is mapped onto a circle not passing through origin.



(vii) A circle passing through origin is mapped onto a straight line not passing through origin.



(viii) A straight line not passing through origin is mapped onto a circle passing through origin

(ix) A straight line passing through origin is mapped onto a straight line passing through origin.

Theorem The mapping $w = 1/z$ maps every straight line or circle onto a circle or straight line.

Proof:

Every straight line in z-plane can be written as

$A(x^2+y^2) + Bx + Cy + D = 0$ ——① where A, B, C, D

are real numbers.

when $A=0$, the above eqn gives a straight line and when $A \neq 0$ it represents a circle.

We have $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

and $z\bar{z} = x^2+y^2$

$|z|^2 = z\bar{z}$

$(\sqrt{x^2+y^2})^2 = z\bar{z}$

$x^2+y^2 = z\bar{z}$

Substituting these in eqn(1), we have

$$Az\bar{z} + B\left(\frac{z+\bar{z}}{2}\right) + C\left(\frac{z-\bar{z}}{2i}\right) + D = 0 \quad \text{—(2)}$$

When $w = 1/z \Rightarrow z = 1/w$, hence $\bar{z} = 1/\bar{w}$

Then eqn(2) becomes,

$$A \times 1/w \times 1/\bar{w} + B\left(\frac{1/w + 1/\bar{w}}{2}\right) + C\left(\frac{1/w - 1/\bar{w}}{2i}\right) + D = 0$$

$$A\,\frac{1}{w\bar{w}} + B\,\frac{\bar{w}+w}{2w\bar{w}} + C\left(\frac{\bar{w}-w}{2i\,w\bar{w}}\right) + D = 0$$

Multiply throughout by $w\bar{w}$ gives,

$$A + B\left(\frac{\bar{w}+w}{2}\right) + C\left(\frac{\bar{w}-w}{2i}\right) + Dw\bar{w} = 0 \text{ —}$$

Since, $w = u+iv$

$$u = \frac{w+\bar{w}}{2}, \quad v = \frac{w-\bar{w}}{2i}, \quad w\bar{w} = u^2+v^2$$

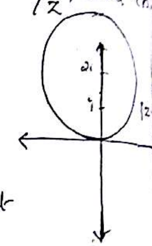
Substituting, we get

$$A + Bu - Cv + D(u^2+v^2) = 0$$

$$D(u^2+v^2) + Bu - Cv + A = 0$$

This eqn represents a circle if $D \neq 0$ & straight line if $D = 0$, in the w-plane.

? Under the transformation $w = 1/z$, Find the image of $|z - 2i| = 2$ (centre $2i$, radius 2)



Ans. $w = 1/z \Rightarrow z = 1/w$ and $\bar{z} = 1/\bar{w}$

$|z - 2i| = 2$

squaring on both sides, we get

$|z-2i|^2 = 4$

$\{ |z|^2 = z\bar{z}$

$(z-2i)(\overline{z-2i}) = 4$

$\{ \overline{z_1 - z_2} = \bar{z}_1 + (\ldots$

$= \bar{z} + 2i$

$(z-2i)(\bar{z}+2i) = 4$

$(1/w - 2i)(1/\bar{w} + 2i) = 4$ (on substituting $z = 1/w$, $\bar{z} = 1/\bar{w}$)

$$\frac{1}{w\bar{w}} + \frac{2i}{w} - \frac{2i}{\bar{w}} - 4i^2 = 4$$

$$\frac{1}{w\bar{w}} + \frac{2i}{w} - \frac{2i}{\bar{w}} = 0$$

$$\frac{1}{w\bar{w}} + 2i\left(\frac{1}{w} - \frac{1}{\bar{w}}\right) = 0$$

$$\frac{1}{w\bar{w}} + 2i\left(\frac{\bar{w}-w}{w\bar{w}}\right) = 0$$

$$\frac{2i(\bar{w}-w)+1}{w\bar{w}} = 0 \Rightarrow 2i\bar{w} - 2iw + 1 = 0$$

$$2i\bar{w} - 2iw + 1 = 0$$

$$1 + 2i(\bar{w} - w) = 0$$

$$1 + 2i(-2iv) = 0$$

$$1 + 4v = 0,$$

$u = \frac{w+\bar{w}}{2}$

$v = \frac{w-\bar{w}}{2i}$

$w - \bar{w} = 2iv$

$\bar{w} - w = -2iv$

which is a straight line in w-plane.

? Find the image of the circle $|z-3| = 5$ (centre, radius) under the transformation $w = 1/z$

Ans. $|z-3| = 5$

Squaring on both side, we get,

$$|z-3|^2 = 5^2 \Rightarrow (z-3)(\overline{z-3}) = 25$$

$$(z-3)(\bar{z}-3) = 25$$

Substituting $z = 1/w$, $\bar{z} = 1/\bar{w}$, we get

$$\left(\frac{1}{w} - 3\right)\left(\frac{1}{\bar{w}} - 3\right) = 25$$

$$\frac{1}{w\bar{w}} - \frac{3}{w} - \frac{3}{\bar{w}} + 9 = 25$$

$$\frac{1}{w\bar{w}} - 3\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) = 16$$

$$\frac{1}{w\bar{w}} - \frac{3(\bar{w}+w)}{w\bar{w}} = 16$$

$$1 - 3\bar{w} - 3w = 16w\bar{w}$$

$$1 - 3(\bar{w} + w) = 16w\bar{w}$$

$$1 - 3(\bar{w} + w) - 16w\bar{w} = 0$$

$$1 - 3(2u) - 16(u^2+v^2) = 0$$

$$1 - 6u - 16(u^2+v^2) = 0$$

$$16(u^2+v^2) + 6u - 1 = 0,$$

$u = \frac{w+\bar{w}}{2}$

$2u = w + \bar{w}$

$w\bar{w} = u^2 + v^2$

$\because w\bar{w} = |w|^2$

$(\sqrt{u^2+v^2})^2 = u^2+v^2$

which represents the eqn of a circle in w-plane.

(iv) The mapping $w = z + 1/z$

mapping is not conformal except $f'(z) = 0$

$f'(z) = 0 \Rightarrow 1 - \frac{1}{z^2} = 0 \Rightarrow z = \pm 1$

Let $z = re^{i\theta}$ and $w = u + iv$

Now, $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$

$$\therefore z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$z + \frac{1}{z} = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Therefore, $u = \left(r + \frac{1}{r}\right)\cos\theta$

$v = \left(r - \frac{1}{r}\right)\sin\theta$

Properties :

- The circle $|z| = c$, $c \neq 1$ is mapped onto ~~the~~ an Ellipse $\frac{u^2}{(c+1/c)^2} + \frac{v^2}{(c-1/c)^2} = 1$, which is an Ellipse in w-plane.

11/26

ie, $u = (c + 1/c)\cos\theta$, $v = (c - 1/c)\sin\theta$

$$u^2 = (c+1/c)^2\cos^2\theta$$

$$v^2 = (c-1/c)^2\sin^2\theta$$

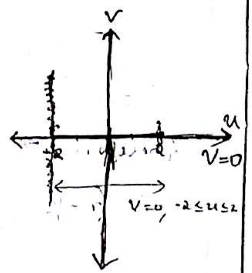
$$u^2/(c+1/c)^2 = \cos^2\theta$$

$$v^2/(c-1/c)^2 = \sin^2\theta$$

- when $c = 1$, ie, $|z| = 1$, the unit circle we have ie, $r = 1$

$$\Rightarrow u = (1 + 1/1)\cos\theta = \underline{2\cos\theta}$$

$$v = (1 - 1/1)\sin\theta = \underline{0}$$



Since, $-1 \leq \cos\theta \leq 1$, we have

$$-2 \leq 2\cos\theta \leq 2$$

$$-2 \leq u \leq 2 \text{ and } v = 0,$$

which is a line segment. or which is the segment of u-axis.

- The Ray $\theta = c$

11/27 Here, $u = (r + 1/r)\cos c$

and $v = (r - 1/r)\sin c$.

$$u^2/\cos^2 c - v^2/\sin^2 c = (r + 1/r)^2 - (r - 1/r)^2$$

$$= r^2 + 1/r^2 + 2 - (r^2 + 1/r^2 - 2) \quad \begin{cases}(a+b)^2 = a^2+b^2+2ab \\ (a-b)^2 = a^2+b^2-2ab\end{cases}$$

$$= 4.$$

$$u^2/\cos^2 c - v^2/\sin^2 c = 4$$

$$u^2/4\cos^2 c - v^2/4\sin^2 c = 1$$, which is ~~the hyperbola~~ a hyperbola with foci $(\pm 2, 0)$ and eccentricity $\sec c$.

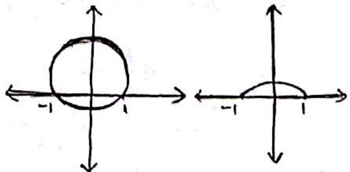
- The mapping $w = z + 1/z$ is not conformal where $w' = 0$

ie, $w' = 1 - 1/z^2 = 0 \Rightarrow z^2 - 1 = 0$,

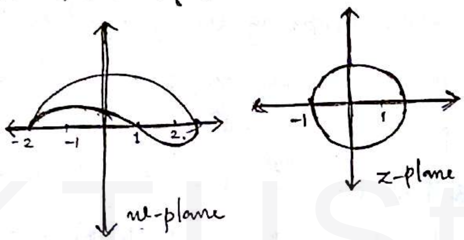
$$z^2 = 1 \Rightarrow z = \pm 1.$$

- A circle passing through both $z=1$ & $z=-1$

are mapped onto a curved ~~surface~~ segment.



- A circle passing through $z = -1$ is mapped onto a Joukowski airfoil.



LINEAR TRANSFORMATION:

A mapping of the form $w = az+b$, where a & b are complex or real constants, is called a linear transformation.

when $a = 1$, the mapping $w = z+b$, which represents a translation. Here, the image of any curve or region in the z-plane is just the translated curve or region in the w-plane.

when $b = 0$, the mapping $w = az$, $a \neq 0$ represents a rotation or magnification (Expansion or contraction)

when $|a| = 1$, it is a rotation.

when $|a| > 1$, it is an expansion.

when $|a| < 1$, it is a contraction.

when $a = |a|e^{i\theta}$, the transformation expands (or contracts) the radius vector representing z by the factor $|a|$ and rotates it through an angle 'Arg a'.

The image of any region in the z-plane under the transformation $w = az$ has the same geometry except that it is expanded or contracted, rotated and translated

Linear Fractional Transformation (Bilinear or Mobius transformation):

A mapping of the form $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ where, a, b, c, d are complex ~~nos~~ or real numbers, is called a linear fractional transformation or bilinear transformation or mobius transformation.

NOTE:

1. Bilinear mappings are conformal.
2. A mobius transformation is a combination of translation, rotation, magnification and inversion.

★ Theorem:

Every linear fractional translation (bilinear translation) maps the totality of circles & straight lines in the z-plane onto the totality of circles and straight lines in the w-plane.

★ INVERSE MAPPING:

The inverse mapping of $w = \frac{az+b}{cz+d}$ is,

$$z = \frac{dw-b}{-cw+a}.$$

$w = \frac{az+b}{cz+d}$

$wcz + dw = az + b$

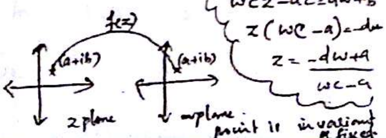
$wcz - az = -dw + b$

$z(wc - a) = -dw + b$

$z = \frac{-dw+b}{wc-a}$

★ FIXED POINTS:

Fixed points of a mapping $w = f(z)$



are the points that are mapped onto themselves. That is, $w = f(z) = z$.

Examples:

(i) The identity mapping $w = z$ has every points a fixed point.

(ii) $w = z - 1$ has no fixed points.

(iii) The mapping $w = 1/z$ has two fixed point $z = \pm 1$

(iv) The mapping $w = \bar{z}$ has infinitely many fixed points. (on the real line)

★ Theorem: (Fixed points of a bilinear transformation)

Any linear fractional transformation, not the identity transformation, has atmost 2 fixed points. If a linear fractional transformation is known to have 3 or more fixed points, it must be the identity mapping $w = z$.

? Find the fixed points of the transformation

$w = 5 - 4z / 4z - 2$

Ans $f(z) = z \Rightarrow z = 5 - 4z / 4z - 2$

$$4z^2 - 2z = 5 - 4z$$

$$4z^2 + 2z - 5 = 0$$

$$z = \frac{-2 \pm \sqrt{4+80}}{8} = \frac{-2 \pm \sqrt{84}}{8}$$

$$\begin{array}{r|l} 2 & 84 \\ 2 & 42 \\ 3 & 21 \\ & 7 \end{array}$$

$$= \frac{-2 \pm 2\sqrt{21}}{8} = \frac{\cancel{2}(-1 \pm \sqrt{21})}{\cancel{8}\,4}$$

$$\therefore z = -\tfrac{1}{4} \pm \tfrac{\sqrt{21}}{4}$$

$\therefore$ fixed points are, $z = -\tfrac{1}{4} + \tfrac{\sqrt{21}}{4}$ and $z = -\tfrac{1}{4} - \tfrac{\sqrt{21}}{4}$

Q? Find the fixed points of the following.

(i) $w = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right)$

(ii) $w = \tfrac{z-1}{z+1}$

(iii) $w = \tfrac{3z-4}{z-1}$

(iv) $w = \tfrac{1}{z - 2i}$

(v) $w = \tfrac{3iz+13}{z-3i}$

Ans. (i) The fixed points are given by, $w = f(z) = z$.

$$z = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right)$$

$$2z = z + \tfrac{1}{z} \Rightarrow 2z - z - \tfrac{1}{z} = 0$$

$$z - \tfrac{1}{z} = 0 \Rightarrow z^2 - 1 = 0$$

$$z^2 = 1 \Rightarrow z = \pm 1$$

(ii) $z = \tfrac{z-1}{z+1}$

$$z(z+1) = z - 1$$

$$z^2 + z = z - 1 \Rightarrow z^2 = -1 \Rightarrow z = \pm\sqrt{-1}$$

$$z = \pm i$$

(iii) $z = \tfrac{3z-4}{z-1}$

$$z^2 - z = 3z - 4 \Rightarrow z^2 - 4z - 4 = 0$$

$$z = \frac{4 \pm \sqrt{16+16}}{2} = \frac{4 \pm \sqrt{32}}{2}$$

$$\begin{array}{r|l} 2 & 32 \\ 2 & 16 \\ 2 & 8 \\ 2 & 4 \\ & 2 \end{array}$$

$$= \frac{4 \pm 4\sqrt{2}}{2} = \frac{\overset{2}{\cancel{4}}(1 \pm \sqrt{2})}{\cancel{2}}$$

$$\therefore z = 2 \pm 2\sqrt{2}$$

(iv) $z = \tfrac{1}{z - 2i}$

$$\cancel{z^2 - 2zi = 1} \quad z(z - 2i) = 1$$

$$z^2 - 2iz - 1 = 0$$

$$\Rightarrow z = \frac{2i \pm \sqrt{4i^2 + 4}}{2} = \frac{2i \pm 0}{2} = i$$

* CROSS RATIO:

Let z_1, z_2, z_3, z_4 be four distinct points in the extended complex plane. The cross ratio of the four points is defined as,

$$[z_1 z_2 z_3 z_4] = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

If any of the point is ∞, then the corresponding factor is deleted from the cross ratio.

Eg :- If $z_1 = \infty$. $[z_1 z_2 z_3 z_4] = \frac{z_2-z_4}{z_2-z_3}$

$z_2 = \infty$ $[z_1 z_2 z_3 z_4] = \frac{z_1-z_3}{z_1-z_4}$.

Theorem :

A bilinear transformation preserves cross ratio.

ie, if w_1, w_2, w_3, w_4 be the images of four distinct points z_1, z_2, z_3, z_4 under the bilinear

? Find the bilinear mapping that maps $z = \overset{z_1}{i}, \overset{z_2}{-i}, \overset{z_3}{1}$ into $w = \overset{w_1}{0}, \overset{w_2}{1}, \overset{w_3}{\infty}$ respectively.

Ans. when $z = i, w = 0$
$z = -i, w = 1$
$z = 1, w = \infty$

→ Let the transformation be $w = \frac{az+b}{cz+d}$, then $[w, 0, 1, \infty] = [z, i, -i, 1]$

The required bilinear mapping is given by,

$$\frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

$$\frac{(w-1)}{0-1} = \frac{(z+i)(i-1)}{(z-1)(i+i)}$$

$$\frac{w-1}{-1} = \frac{(z+i)(i-1)}{2i(z-1)}$$

ie, $$w = \frac{-1(z+i)(i-1)}{2i(z-1)} + 1$$

$$w = \frac{(-z-i)(i-1)}{2i(z-1)} + 1$$

$$= \frac{-iz + z - i^2 + i}{2i(z-1)} + 1 = \frac{i(-z+1) + (z+1)}{2i(z-1)}$$

$$= \frac{-iz + z + 1 + i + 2iz - 2i}{2iz - 2i}$$

$$= \frac{i\cancel{z} + z + 1 - i}{2iz - 2i} = \frac{z(1+i) + 1 - i}{2iz - 2i}$$

$$w = \frac{(1+i)z + (1-i)}{2iz - 2i}, \quad \frac{az+b}{cz+d} \Rightarrow a = 1+i,\ b = 1-i,\ c = 2i,\ d = -2i$$

checking :
when $z = i$

$$w = \frac{(1+i)i + (1-i)}{2i\,i - 2i}$$

$$= \frac{i^2 + \cancel{i} + 1 - \cancel{i}}{-2 - 2i} = \frac{0}{-2-2i} = 0$$

? Find the linear fractional transformation (LFT) that maps the given 3 points onto the 3 given points in the respective order.

(i) $-1, 0, 1$ onto $-1, -i, 1$

(ii) $0, 1, \infty$ onto $-1, -i, 1$

(iii) $-1, i, 1$ onto $0, 1, \infty$

(iv) $-2, 0, 2$ onto $\infty, 1/4, 3/8$

(v) $-i, 0, i$ onto $-1, i, 1$

(vi) $2, i, -2$ onto $1, -i, -1$

(vii) $0, 1, 2$ onto $1, 1/2, 1/3$

(viii) $1, i, -1$ onto $i, -1, -i$

(ix) $0, -i, 1$ onto $-1, 0, \infty$

(x) $-1, 0, 1$ onto $-i, -1, i$

(xi) $0, 1, \infty$ onto $\infty, 1, 0$

(viii) Ans. $[z, 1, i, -1] = [w, i, -1, -i]$

$$\frac{(w-i)(1+1)}{(w+1)(1-i)} = \frac{(z-1)(i+1)}{(z+i)(i-1)}$$

$$\frac{(w-i)\,2}{(w+1)(1-i)} = \frac{(z-1)\,2i}{(z+i)(i+1)}$$

$$\frac{2w - 2i}{w - wi + 1 - i} = \frac{2iz - 2i}{zi + z + i^2 + i}$$

$$\frac{(w-1)(i-i)}{(w-i)(i-1)} = \frac{(z-i)(1-1)}{(z+1)(1-i)}$$

$$\frac{w+1}{(w-i)(i+1)} = \frac{(z-i)\,2}{(z+1)(1-i)}$$

RISWANA

$$\cancel{(w+1)(z+1)(1-i)} = \cancel{2(z-i)(w-i)(i+1)}$$

Answers.

1) $w = \frac{z-i}{-iz+1}$

2) $w = \frac{z-i}{z+i}$

3) $w = -\frac{z+1}{z-1}$

4) $w = \frac{z+1}{2z+4}$

5) $w = \frac{iz+i}{z+1}$

(i) $[z,-1,0,1] = [w,-1,-i,1]$

$$\frac{(w-i)(-1-1)}{(w-1)(-1+i)} = \frac{(z-0)(-1-1)}{(z-1)(-1-0)}$$

$$\frac{(w+i)\cancel{-2}}{(w-1)(-1+i)} = \frac{\cancel{-2}z}{-1(z-1)}$$

$$\frac{(w+i)}{-w+iw+1-i} = \frac{z}{1-z}$$

$$(w+i)(1-z) = -zw+ziw+z-iz$$

$$w\cancel{-zw}+i\cancel{-iz} = \cancel{-zw}+ziw+z\cancel{-iz}$$

$$w+i = ziw+z$$

$$w-ziw = z-i$$

$$w(1-zi) = z-i$$

$$\therefore\ w = z-i/-iz+1$$

(ii) $[z,0,1,\infty] = [w,-1,-i,1]$

$$\frac{(w+i)(-1-1)}{(w-1)(-1+i)} = \frac{(z-1)}{(0-1)}$$

$$\frac{-2(w+i)}{(w-1)(-1+i)} = 1-z$$

$$-2w-2i = (-w+wi+1-i)(1-z)$$

$$-2w-2i = -w+wz+wi-wiz+1-z-i+iz$$

$$\cancel{-2w+w = wz+wi-wiz+1-z-i+iz+2i}$$

$$\cancel{-w = wz-z+1+i(w-2z)}$$

$$-2w+w-wz-wi+wiz = 2i+1-z-i+iz$$

$$w(-2+1-z-i+iz) = 2i+1-z-i+iz$$

$$w = \frac{i+1-z+iz}{-1-z-i+iz} = \frac{i(z+1)+(1-z)}{i(z-1)+(-1-z)}$$

$$= \frac{z(-1+i)+(i+1)}{z(-1+i)+(-i-i)}$$

$-z+1$

$$w = \frac{z-i}{z+i}$$

(iii) $z,-1,i,1] = [w,0,1,\infty]$

$$\frac{(w-1)}{(0-1)} = \frac{(z-i)(-1-1)}{(z-1)(-1-i)}$$

$$w-1 = \frac{(i-z)(-2)}{(z-1)(-1-i)}$$

$$w = \frac{-2i+2z}{-z-zi+1+i} + 1$$

$$= \frac{-2i+2z-z-zi+1+i}{-z-zi+1+i}$$

$$= \frac{-i+z-zi+1}{-z-zi+1+i}$$

$$= \frac{z(1-i)+(1-i)}{z(-1-i)+(1+i)}$$

$$= \frac{(1-i)(z+1)}{(1+i)(-z+1)}$$

$$\left\{ \frac{(1-i)}{(1+i)}\frac{(1-i)}{(1-i)} \quad \frac{1-i-i+i^2}{1^2-i^2} = \frac{-2i}{2} \right.$$

(iv) $[z,-2,0,2] = [w, \infty, 1/4, 3/8]$

$$\frac{(w-1/4)}{(w-3/8)} = \frac{(z-0)(-2-2)}{(z-2)-2}$$

$$(w-1/4)(z-2)\cancel{-2} = (w-3/8)\overset{2}{\cancel{-4}}z$$

$$wz-2w-1/4z+1/2 = 2zw-3/4z$$

$$wz-2w-2zw = -3/4z+1/4z-1/2$$

$$w(z-2-2z) = -1/2z-1/2$$

$$w = \frac{-1/2z-1/2}{z-2-2z} = \frac{-1/2z-1/2}{-z-2}$$

$$w = \frac{1/2z+1/2}{z+2}$$

(v) $[z,-i,0,i] = [w,-1,i,1]$

$$\frac{(w-i)(-1-1)}{(w-1)(-1-i)} = \frac{(z-0)(-i-i)}{(z-i)(-i-0)}$$

$$\frac{\cancel{-2}(w-i)}{(w-1)(-1-i)} = \frac{\cancel{-2}iz}{(z-i)-i}$$

$$(w-i)(-zi+\overset{-1}{i^2}) = (-w-wi+1+i)iz$$

$$\cancel{-wzi}-w+zi^2+i = \cancel{-wiz}-wi^2z+iz+i^2z$$

$$-w\cancel{-z}+i = wz\cancel{-z}+iz$$

$$-w-wz = iz-i$$

$$w(-1-z) = iz-i$$

$$\therefore w = \frac{iz-i}{-1-z}$$

(vi) $[z, 2, i, -2] = [w, 1, -i, -1]$

$$\frac{(w+i)(1-1)}{(w+1)(1+i)} = \frac{(z-i)(2-2)}{(z+2)(2-i)}$$

$$\frac{2(w+i)}{w+wi+1+i} = \frac{4(z-i)}{2z-iz+4-2i}$$

$$(w+i)(2z-iz+4-2i) = (2z-2i)(w+wi+1+i)$$

$$\cancel{2zw} - izw + 4w \cancel{-2wi} + \cancel{2iz} + z + 4i + \cancel{2} =$$
$$\cancel{2zw} + 2zwi + 2z + \cancel{2zi} - \cancel{2iw} + 2w - 2i + \cancel{2}$$

$$-izw + 4w - 2zwi - 2w = -z - 4i + 2z - 2i$$

$$w(-iz + 4 - 2zw - 2) = z - 6i$$

$$\therefore w = \frac{z-6i}{-2zw - iz + 2}$$

$$\therefore w = \frac{z-6i}{z(-2w-i)+2}$$

(vii) $[z, 0, 1, 2] = [w, 1, 1/2, 1/3]$

$$\frac{(w-1/2)(1-1/3)}{(w-1/3)(1-1/2)} = \frac{(z-1)}{(z-2)\,-1}$$

$$\frac{(w-1/2)\,2/3}{(w-1/3)\,1/2} = \frac{z-1}{-z+2}$$

$$(2/3\,w - 1/3)(-z+2) = (1/2\,w - 1/6)(z-1)$$

$$-2/3\,wz + 4/3\,w + z/3 - 2/3 = 1/2\,wz - 1/2\,w - z/6 + 1/6$$

$$w(-2/3\,z + 4/3 - 1/2\,z + 1/2) = -z/3 + 2/3 - z/6 + 1/6$$

$$w = \frac{z(-1/3 - 1/6) + (2/3 + 1/6)}{z(-2/3 - 1/2) + (1/2 + 4/3)}$$

$-9/18 = -1/2$

$\cancel{15/18}$

$$= \frac{z \times -1/2 + 5/6}{z \times -7/6 + 11/6}$$

$$w = \frac{-1/2\,z + 5/6}{-7/6\,z + 11/6}$$

(ix) $[z, 0, -i, 1] = [w, -1, 0, \infty]$

$$\frac{(w-0)}{(-1-0)} = \frac{(z+i)(-1)}{(z-1)(0+i)}$$

$$-w = \frac{-z-i}{(z-1)i}$$

$$w = \frac{z+i}{zi-i}$$

$$\therefore w = \frac{z+i}{zi-i}$$

(x) $[z, -1, 0, 1] = [w, -i, -1, i]$

$$\frac{(w+1)(-i-i)}{(w-i)\,(-i+1)} = \frac{(z-0)(-2)}{[z-1)(-1)}$$

$$\frac{(w+1)\cancel{-2}i}{-wi+w-1-i} = \frac{z\cancel{(-2)}}{-z+1}$$

$$(wi+i)(-z+1) = -zwi+zw-z-zi$$

$$-wiz+iw-iz+i = -zwi+zw-z-zi$$

$$w(\cancel{-iz}+i+\cancel{z}i-z) = -z\cancel{-zi}-i\cancel{+iz}$$

$$w = \frac{(-z-i)}{(-z+i)}$$

(xi) $[z, 0, 1, \infty] = [w, \infty, 1, 0]$

$$\frac{(w-1)}{w} = \frac{(z-1)}{(0-1)}$$

$$(-w+1) = wz-w$$

$$\cancel{w(1}\ \cancel{-w}+\cancel{w}-wz = -1$$

$$w = \frac{\cancel{-}1}{\cancel{-}z}$$

$$\therefore w = 1/z$$

(vii) $[z, 0, 1, 2] = [w, 1, 1/2, 1/3]$

$$\frac{(w-1/2)(1-1/3)}{(w-1/3)(1/2)} = \frac{(z-1)(0-2)}{(z-2)(-1)}$$

$$\frac{\cancel{2}/3\,(w-1/2)}{1/2\,(w-1/3)} = \frac{-\cancel{2}(z-1)}{-z+2}$$

$$(w-1/2)(-z+2) = (-z+1)\ 3/2\,(w-1/3)$$

$$-wz+2w+1/2\,z-1 = (3/2\,w-1/2)(-z+1)$$

$$-wz+2w\cancel{+1/2\,z}-1 = -3/2\,wz+3/2\,w\cancel{+1/2\,z}-1/2$$

$$w(-z+2+3/2\,z-3/2) = -1/2+1$$

$$w\left(\tfrac{1}{2}z+\tfrac{1}{2}\right)=\tfrac{1}{2}$$

$$\therefore w=\frac{1/2}{\tfrac{1}{2}z+\tfrac{1}{2}}$$

(viii) $[z, \infty, i, -1] = [w, i, -1, -i]$

$$\frac{(w+1)(i+i)}{(w+i)(i+1)}=\frac{(z-i)\ (1+1)}{(z+1)\ (1-i)}$$

$$\frac{\cancel{2}i(w+1)}{(w+i)(i+1)}=\frac{(z-i)\cancel{2}}{(z-zi+1-i)}$$

$$(wi+i)(z-zi+1-i)=(wi+w+i^2+i)(z-i)$$

$$wiz-wzi^2+wi-wi^2+zi-zi^2+i\cancel{-i^2}=$$

$$wiz-wi^2+wz-wi+i^2z-i^3+iz\cancel{-i^2}$$

$$\cancel{wiz}+\cancel{wz}+wi+\cancel{w}+\cancel{zi}+z+\cancel{i}=\cancel{wiz}+\cancel{w}+\cancel{wz}-wi-z+\cancel{i}+\cancel{iz}$$

$$wi+z=-wi-z$$

$$2wi=-2z$$

$$\therefore w=\frac{-2z}{2i}\Rightarrow w=\frac{-z}{i}$$

★ The mapping $w=\sin z$

$f(z)=\sin z$

$f'(z)=\cos z$

$\cos z=0\Rightarrow z=(2n+1)\pi/2,\ n=0,\pm1,\pm2,\dots$

$\therefore$ It is not conformal at $n=0,\pm1,\pm2\dots$

The mapping $w=\sin z$ is not conformal where $\cos z=0$ ie, $z=(2n+1)\pi/2,\ n=0,\pm1,\pm2\dots$

$\rightarrow w=\sin z$

$w=\sin(x+iy)$

$\quad=\sin x\cos(iy)+\cos x\sin(iy)$

$\{\cos(iy)=\cosh y,\quad \sin(iy)=i\sinh y\}$

ie, $w=\sin x\cdot\cosh y+i\cos x\sinh y$

ie, $u=\sin x\cosh y,\quad v=\cos x\sinh y$

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad \cos z=\frac{e^{iz}+e^{-iz}}{2}$$

$$\sinh z=\frac{e^{z}-e^{-z}}{2},\quad \cosh z=\frac{e^{z}+e^{-z}}{2}$$

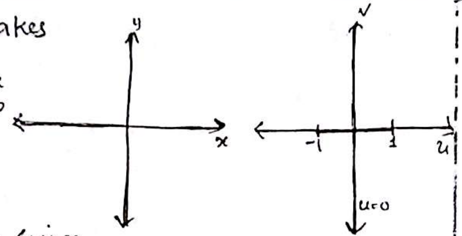
$$\therefore \cos(iy)=\frac{e^{i(iy)}+e^{-i(iy)}}{2}=\frac{e^{-y}+e^{y}}{2}=\cosh y$$

$\sin(iy)=i\sinh y$

Properties:

when $x=0,\ u=0,\ v=\sinh y$

Since, $\sinh y$ takes all real values, the image of the line $x=0$ will be $u=0$. (ie, v-axis)



(2) when $y=0$, $u=\sin x$

$v=0$. Since, $|\sin x| \le 1$, $-1 \le \sin x \le 1$, the image of the line $y=0$ ie, $v=0$ and $|u| \le 1$, which is the line segment $-1 \le u \le 1$.

(3) when $x=c$, $c \neq 0$

$u = \sin c \cosh y$ & $v = \cos c \sinh y$.

then $u^2 = \sin^2 c \cosh^2 y$ & $v^2 = \cos^2 c \sinh^2 y$

$u^2/\sin^2 c = \cosh^2 y$ $\quad$ $v^2/\cos^2 c = \sinh^2 y$

ie $\cosh^2 y - \sinh^2 y = u^2/\sin^2 c - v^2/\cos^2 c = 1$

$u^2/\sin^2 c - v^2/\cos^2 c = 1$, which is a hyperbola. Hence, the image of the line $x=c$ is the hyperbola $u^2/\sin^2 c - v^2/\cos^2 c = 1$ with foci $(\pm 1, 0)$, and eccentricity $\operatorname{cosec} c$.

(4) when $y=c$, $c \neq 0$

$u = \sin x \cosh c$, $\quad v = \cos x \sinh c$

then, $u^2 = \sin^2 x \cosh^2 c$, $\quad v^2 = \cos^2 x \sinh^2 c$

$\dfrac{u^2}{\cosh^2 c} = \sin^2 x$, ...(1) $\quad \dfrac{v^2}{\sinh^2 c} = \cos^2 x$

(1) + (2) $\Rightarrow \dfrac{u^2}{\cosh^2 c} + \dfrac{v^2}{\sinh^2 c} = \sin^2 x + \cos^2 x = 1$

ie, $\dfrac{u^2}{\cosh^2 c} + \dfrac{v^2}{\sinh^2 c} = 1$, which is an ellipse.

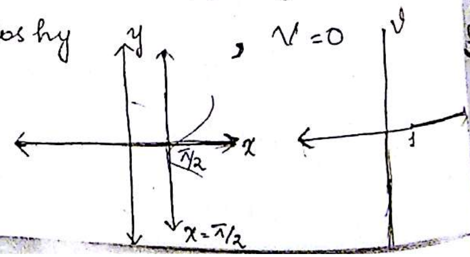
Hence, the image of the line $y=c$ is the ellipse $u^2/\cosh^2 c + v^2/\sinh^2 c = 1$, with foci $(\pm 1, 0)$ and eccentricity $\operatorname{sech} c$.

(5) when $x = \pi/2$

Then, $u = \sin x \cosh y$, $v = \cos x \sinh y$

when $x = \pi/2$, $u = \sin \pi/2 \cosh y$

$u = \cosh y$, $\quad v = 0$



we have $\cosh y \geq 1$

ie, $u \geq 1$ and $v = 0$

$\therefore$ the image of the line $x = \pi/2$ is $u \geq 1$. (say)

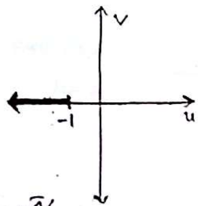
(6) when $x = -\pi/2$:

$u = -\cosh y$, $v = 0$

we have $\cosh y \geq 1$

$\therefore -\cosh y \leq -1$

$\therefore u \leq -1$, Also $v = 0$



Hence, the image of the line $x = -\pi/2$ is the portion of u-axis, $u \leq -1$.

★ The mapping / transformation $w = \cos z$

we have $w = \cos z$

$$w = \cos(x+iy)$$
$$= \cancel{\cos x + i \cos y}$$
$$= \cos x \cos(iy) - \sin x \sin(iy)$$
$$= \cos x \cosh y - i \sin x \sinh y$$

ie, $u = \cos x \cosh y$, $v = -\sin x \sinh y$.

The mapping $w = \cos z$ is not conformal when $-\sin z = 0$, ie, $\sin z = 0$ ie, $z = 0, \pm\pi, \pm 2\pi \ldots$

Since $\cos z = \sin(z + \pi/2)$, $w = \cos z$ is the same transformation as $w = \sin z$; proceeded by a translation to the right through $\pi/2$ units.

$\Rightarrow$ Properties:

(i) When $x = 0$

$u = \cosh y$ and $v = 0$

we have $\cosh y \geq 1$, ie, $u \geq 1$, and $v = 0$.

The image is u-axis with $u \geq 1$.

(ii) when $y = 0$

$u = \cos x$ and $v = 0$

$-1 \leq \cos x \leq 1$

ie, $-1 \leq u \leq 1$ and $v = 0$

Hence, the ~~image~~ is the line segment $-1 \leq u \leq 1$

(iii) when $x = c$, $c \neq 0$

$u = \cos c \cosh y$, $v = -\sin c \sinh y$,

Hence, the image is $u^2/\cos^2 c - v^2/\sin^2 c = 1$, which is a hyperbola with focii $(\pm 1, 0)$ and eccentricity $\sec c$.

(iv) when $y = c$, $c \neq 0$

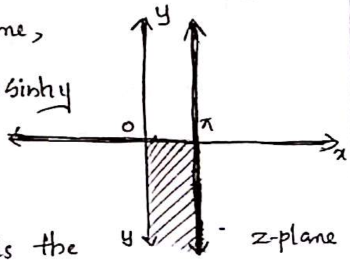
Here, $u = \cos x \cosh c$, $v = -\sin x \sinh c$

The image is the ellipse $u^2/\cosh^2 c + v^2/\sinh^2 c = 1$

with foci $(\pm 1, 0)$ and eccentricity sech c.

? Find the image of the strip $0 \le x \le \pi$, $y < 0$ under the transformation $w = \cos z$. Also, sketch the region.

Ans: In z-plane or x-y plane,



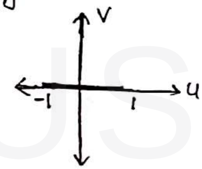
$u = \cos x \cosh y$ & $v = -\sin x \sinh y$

→ when $y = 0$;

$u = \cos x$, $v = 0$

when $y = 0$, the image is the line, $-1 \le u \le 1$.

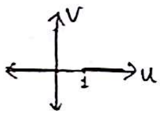
ie, the image is,



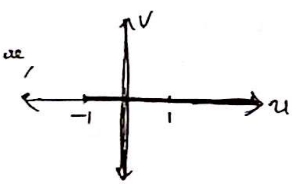
→ when $x = 0$,

$u = \cosh y$, $v = 0$

The image of $x = 0$ is $u \ge 1$. ie,



ie,



→ when $x = \pi$, $u = -\cosh y$, $v = 0$

~~$u \ge 1$ ⇒ $-u \le -1$ ⇒ $-\cosh y \le -1$.~~

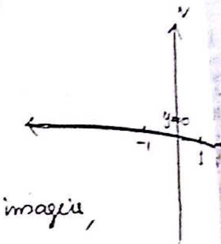
ie,

$\cosh y \ge 1$

hence, $-\cosh y \le -1$

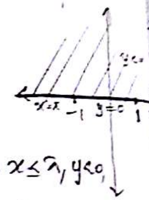
ie, $u \le -1$

ie, image is,



→ when $y < 0$, $\sinh y < 0$, hence

$v = -\sin x \sinh y > 0$ ie,



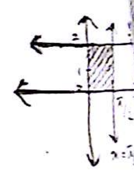
Hence, the image of the strip, $0 \le x \le \pi$, $y < 0$, is the upper half plane in the w-plane.

? Find and sketch the image of the region $\le \pi/2$, $0 < y < 2$, under the transformation $w = \sin z$.

Ans. when $x = 0$,

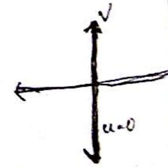
$u = \sin x \cosh y$, $v = \cos x \sinh y$

$$\left\{ \begin{array}{l} x = 0 \\ y = 0 \\ x = \pi/2 \\ y = 2 \end{array} \right.$$

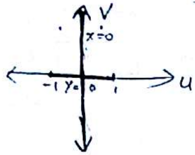


$u = 0$, $v = \sinh y$.

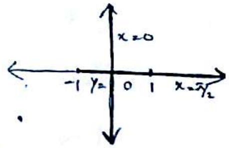
The image of the line $x = 0$ is the line $u = 0$ (ie v-axis)



when $y=0$, the image is the line segment $-1 \le u \le 1$.



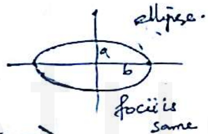
→ Now, when $x = \pi/2$, the image is $u \ge 1$.

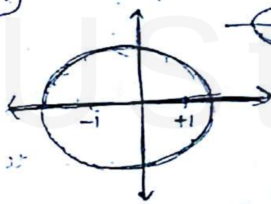


→ Now, when $y = c$,
the image of the line $y=2$ is the ellipse $\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$

ie, $\frac{u^2}{\cosh^2(2)} + \frac{v^2}{\sinh^2(2)} = 1$

with focii $(\pm 1, 0)$





MODULE - 3 ①

COMPLEX INTEGRATION

★ Some basic definitions:

(i) Line Integrals:

Complex definite integrals are called line integrals. They are of the form, $\int_C f(z)dz$ or $\oint_C f(z)dz$ where, C is a given curve called path of integral.

(ii) Curve:

The parametric equation $z(t) = x(t) + i\,y(t)$, $a \le t \le b$, defines a curve in the complex plane. The direction of increasing t is called positive direction on C and C is said to be an oriented curve.

(iii) Smooth curve:

If a curve 'C' has continous and non-zero derivatives at each point, then C is called a smooth curve. (Sharp edge - 2 curves are not smooth)

(iv) Simple curve:

A curve is simple if it doesn't intersect itself. A simple curve which is closed is called a simple closed curve.