Maths

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MODULE-1 Anjali

COMPLEX DIFFERENTIATION

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COMPLEX FUNCTIONS:

eg: 1) $f(z) = z^2$

2) $f(z) = 3z^2 + 2z$

3) $f(z) = e^z$

4) $f(z) = \log z$

5) $f(z) = 2z^2 + 3\bar{z}$

6) $f(z) = z^2 + 2\,Re(z)$

Now, 1) $f(z) = z^2$

$f(i) = i^2 = -1$ Similarly, $f(1+i) = (1+i)^2 = 1 + i^2 + 2i$

$\therefore f(1+i) = 2i$

Now, 2) $f(z) = 3z^2 + 2z$

$f(2+3i) = 3(2+3i)^2 + 2(2+3i)$

→ $f : S \to \mathbb{C}$

here, S is called domain

→ $w = f(z) = u + iv$

$u \to$ real part, $v \to$ imaginary part

If $z = x+iy$

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

→ $f(z) = z^2$

ie, $f(x+iy) = (x+iy)^2 = x^2 + i^2y^2 + 2xyi$

$f(x+iy) = x^2 - y^2 + 2xyi$

∴ $Re[f(x+iy)] = x^2 - y^2$

Similarly, $Im[f(x+iy)] = 2xy$

→ $f(z) = e^z$

ie, $f(x+iy) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$Re[f(x+iy)] = e^x \cos y$

$Im[f(x+iy)] = e^x \sin y$

★ Every Complex function $f(z)$ has two parts. Real and Imaginary parts.

⇒ COMPLEX FUNCTIONS :

Let 'S' be a subset of complex numbers. A function 'f' defined on 'S' is a rule that assigns to every 'z' in 'S' a complex number 'w', which is the value of f at z and we write,

$$w = f(z)$$

The set 'S' is called the domain of f. And the set of all values of the function 'f' is called the range of f.

If $w = f(z)$, where $z = x+iy$, then w can written as, $w = u(x,y) + i\,v(x,y)$.

where u and v are called the real and imaginary parts of $f(z) = w$. Thus a complex function is equalent to a pair of real functions $u(x,y)$ and $v(x,y)$.

? Find the value of the following functions at the given point.

(1) $f(z) = z^2 + 3z$ at $z = 2+i$

(2) $f(z) = 2iz + 6\bar{z}$ at $z = i$

(3) $f(z) = e^z$ at $z = 1$

(4) $f(z) = z^2 - (2+i)z$ at $z = 1-i$

Ans (1) $f(z) = z^2 + 3z$ at $z = 2+i$

ie, $f(2+i) = (2+i)^2 + 3(2+i)$

$= 4 + i^2 + 4i + 6 + 3i$

$= 10 - 1 + 7i$

$f(2+i) = \underline{\underline{9+7i}}$

(2) $f(z) = 2iz + 6\bar{z}$ at $z = i$

ie $f(i) = 2i \times i + 6 \times \bar{i}$ $\left\{ \begin{array}{l} z = i \\ \bar{z} = -i \end{array} \right.$

$= 2i^2 + -6i = -2 - 6i$

(3) $f(z) = e^z$ at $z = 1$.

ie $f(1) = e^1 = e$.

(4) $f(z) = z^2 - (2+i)z$ at $z = 1-i$

$f(1-i) = (1-i)^2 - (2+i)(1-i)$

$= 1 + i^2 - 2i - (2 - 2i + i - i^2)$

$= 1 - 1 - 2i - 2 + 2i - i + i^2$

$= -2 - i - 1 = -3 - i$

$\therefore f(1-i) = -3 - i$

(5) $f(z) = z + 2\,Re(z)$ at $z = 3+4i$

ie, $f(3+4i) = 3 + 4i + 2 \times 3 = 3 + 4i + 6$

$f(3+4i) = 9 + 4i$

? Find the real and imaginary parts of the following functions

(1) $f(z) = z^2 + 3z$

(2) $f(z) = z^2$

(3) $f(z) = e^z$

(4) $f(z) = z^2 + (2+i)z$

(5) $f(z) = \log z$.

Ans. (1) $f(z) = z^2 + 3z$

$f(x+iy) = (x+iy)^2 + 3(x+iy)$

$= x^2 + -y^2 + 2xyi + 3x + 3yi$

$f(x+iy) = x^2 - y^2 + 3x + i(2xy + 3y)$

$\therefore$, $Re\, f(z) = Re[f(x+iy)] = x^2 - y^2 + 3x = u(x,y)$.

$Im\, f(z) = 2xy + 3y = v(x,y)$.

(2) $f(z) = z^2$

$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2xyi$

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

(3) $f(z) = e^z$

$f(x+iy) = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$f(x+iy) = e^x \cos y + i\, e^x \sin y$

$u(x,y) = e^x \cos y$

$v(x,y) = e^x \sin y$

(4) $f(z) = z^2 + (2+i)z$

$f(x+iy) = (x+iy)^2 + (2+i)(x+iy)$

$= x^2 - y^2 + 2xyi + 2x + 2iy + ix - y$

$= (x^2 - y^2 + 2x - y) + i(2x + 2y + x)$

$u(x,y) = x^2 - y^2 + 2x - y$

$v(x,y) = 2x + 2y + x$

(5) $f(z) = \log z$

$f(x+iy) = \log(x+iy)$

$f(z) = \log z$, put $z = re^{i\theta}$

$= \log(re^{i\theta}) = \log r + \log e^{i\theta}$

$= \log r + i\theta$

$f(z) = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$

$\therefore u(x,y) = \log\sqrt{x^2+y^2}$ $\quad \{ \log z = \log|z| + i\,\mathrm{Arg}\,z$

$v(x,y) = \tan^{-1}(y/x)$

? Find the real and Imaginary.

(1) $f(z) = \bar{z}$

(2) $f(z) = 1/z$

(3) $f(z) = z^3$

Ans. (1) $f(z) = \bar{z}$

$f(x+iy) = \overline{x+iy} = x - iy$

$\therefore u(x,y) = x$; $v(x,y) = -y$

(2) $f(z) = 1/z$

$f(x+iy) = \frac{1}{(x+iy)} \times \frac{(x-iy)}{(x-iy)}$

$= \frac{x-iy}{x^2 - i^2y^2} = x - iy/x^2+y^2$

$f(x+iy) = x/_{x^2+y^2} - i(y/_{x^2+y^2})$

$\therefore u(x,y) = x/_{x^2+y^2}$, $v(x,y) = -(y/_{x^2+y^2})$

(3) $f(z) = z^3$

$f(x+iy) = (x+iy)(x+iy)(x+iy) = (x+iy)^3$

$= x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3x^2iy + -3xy^2 + y^3 \times -i$

$f(x+iy) = x^3 - 3xy^2 + i(3x^2y - y^3)$

$\therefore u(x,y) = x^3 - 3xy^2$

Similarly, $v(x,y) = 3x^2y - y^3$

⇒ LIMITS & CONTINUITY:

A function $f(z)$ is said to have the limit 'l' as z approaches a point z_0. If z is defined in a neighbourhood of z_0 and the values of f are close to 'l' for all z close to z_0. $\left\{\begin{matrix} f(z) \cdots l \\ z \cdots z_0 \end{matrix}\right.$

In otherwords, for every positive real Epsilon, we can find a positive real delta such that for all $z \neq z_0$, $|z-z_0| < \delta$, we have $|f(z)-l| < \varepsilon$

* A function $f(z)$ is said to be continous at $z = z_0$ if $f(z_0)$ is defined and $\lim_{z\to z_0} f(z) = f(z_0)$

$f(z)$ is continous on a domain, if it is continous at each point of this domain.

? $f(z) = \sin z$

Ans. $f(z) = \sin z$

$f(z) = \sin(x+iy)$

$= \sin x \cos(iy) + \cos x \sin(iy)$

$= \sin x \cosh z + \cos x\, i \sinh z$

$f(z) = \sin x \cosh z + i(\cos x \sinh z)$

∴ $u(x,y)$ and $v(x,y)$

$\left\{\begin{array}{l} e^{iz} = \cos z + i\sin z \\ e^{-iz} = \cos z - i\sin z \\ e^{iz}+e^{-iz} = 2\cos z \\ \cos z = \dfrac{e^{iz}+e^{-iz}}{2} \\ e^{iz}-e^{-iz} = 2i\sin z \\ \sin z = \dfrac{e^{iz}-e^{-iz}}{2i} \\ \cos(iz) = \cosh z \\ \sin(iz) = i\sinh z \end{array}\right.$

e^{iz}, e^{-iz} ; $\dfrac{e^{z}+e^{-z}}{2}$; $\cosh x = \dfrac{e^{x}-e^{-x}}{2}$; $\dfrac{e^{-z}-e^{z}}{2i}$

⇒ Derivative of $f(z)$:

$f'(z_0) = \lim_{\Delta z\to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

$\left\{\begin{array}{l} \lim_{z\to z_0} f(z) = l, \\ \lim_{z\to z_0} f(z) = f(z_0) \end{array}\right.$

If $z - z_0 = \Delta z$

$z = z_0 + \Delta z$

∴ $f'(z_0) = \dfrac{f(z)-f(z_0)}{z-z_0}$

The derivative of the complex function $f(z)$ at the point z_0 written as $f'(z_0)$ is defined as,

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

eg: 1) The function $f(z) = z^2$ is differentiable at all z

2) The function $f(z) = \bar{z}$ is no where differentiable.

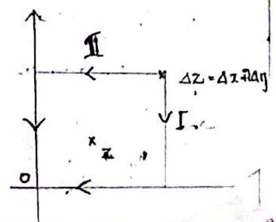
3) The function $f(z) = |z|$ is differentiable only at $z = 0$

? Show that the function $f(z) = \bar{z}$ is no where differentiable.

Ans. $f(z) = \bar{z}$, $z = x+iy$

$\Delta z = \Delta x + i\Delta y$

$f'(z) = \lim_{\Delta z\to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$



$$= \lim_{\Delta z \to 0} \frac{\overline{z_0+\Delta z} - \bar{z}_0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{\bar{z}_0} + \overline{\Delta z} - \cancel{\bar{z}_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta z \to 0$ along a line parallel to X-axis, then $\Delta y = 0$ and Δz becomes Δx. In this path,

$$\lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = 1.$$

If $\Delta z \to 0$ along a line parallel to y-axis, then $\Delta x = 0$ and hence $\Delta z = i\Delta y$. In this path,

ie, $$\lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta z \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Therefore, we can say that the limit does not exist. Since the two limits are different;

ie, $f(z) = \bar{z}$ is not differentiable.

ANALYTIC FUNCTIONS:

→ A function $f(z)$ is said to be analytic at a point z_0 if $f(z)$ is differentiable at z_0 and in some neighbourhood of z_0.

→ A function $f(z)$ is said to be analytic in a domain 'D' if it is analytic in all points in D.

Entire function:

A fun $f(z)$ that is analytic at every point z [in] the complex plane is said to be an entire function.

Singular points:

A point at which a complex fun, $w = f(z)$ fails to be analytic is called a singular point of $f(z)$.

eg: 1) $z=0$ is a singular point of $f(z) = 1/z$

2) $z = i$ is a singular point of $f(z) = 1/(z^2+1)$

NOTE:

1. The sum, product, composition and quotient of analytic functions are analytic.
2. A polynomial fun of the form $f(z) = c_0 + c_1 z + c_2 z^2 + \dots\dots c_n z^n$ is analytic at every point in the complex function.
3. A rational function of the form $f(z) = \frac{g(z)}{h(z)}$ is analytic at all points except at the points where $h(z) = 0$.
4. The complex exponential fun $f(z) = e^z$ is analytic everywhere.

eg: 3) The fun $f(z) = \frac{4z}{z^2 - 2z + 2}$ is not analytic when $z^2 - 2z + 2 = 0$

$\Rightarrow z = 1 \pm i$

$\therefore$, $f(z)$ is not analytic at $z = 1+i$ & $z = 1-i$.

In other words, these points are the singular points of analytic function.

⇒ CAUCHY – RIEMANN EQUATIONS (C-R eqns):

If $w = f(z) = u(x,y) + iv(x,y)$ is an analytic fun, then u and v satisfies the Cauchy Riemann eqn

$u_x = v_y$ and $u_y = -v_x$

$u_x = v_y$
$u_y = -v_x$

eg: $f(z) = z^2$.

$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u(x,y) = x^2 - y^2$

$v(x,y) = 2ixy$

Now, $u_x = 2x$ $\quad u_y = -2y$

$v_x = 2y$ $\quad v_y = 2x$

from this, $u_x = v_y$ and $u_y = -v_x$

Hence, Cauchy Reimann eqns are satisfied.

• $f(z) = \bar{z}$

$f(z) = \bar{z} = (\overline{x+iy}) = x - iy$

$u(x,y) = x$, $\quad v(x,y) = -y$.

$u_x = 1$ $\quad u_y = 0$ $\quad v_x = 0$ $\quad v_y = -1$

here, $u_x \neq v_y$.

Therefore, Cauchy reimann eqn are not satisfied.

hence, $f(z) = \bar{z}$ is not analytic

So, it is not differentiable.

• $f(z) = e^z$

$f(z) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

here, $u(x,y) = e^x \cos y$; $v(x,y) = e^x \sin y$

Now, $u_x = \cos y\, e^x$ $\quad u_y = -e^x \sin y$

$v_x = \sin y\, e^x$ $\quad v_y = e^x \cos y$

ie, $u_x = v_y$ and $u_y = -v_x$

Hence, Cauchy reimann eqns are satisfied.

therefore, $f(z) = e^z$ is analytic.

Theorem:

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighbourhood of a point $z = x+iy$ and differentiable at z itself. Then, at that point the first order partial derivative of u and v exist and satisfies the Cauchy-Reimann eqn $u_x = v_y$, $u_y = -v_x$

Proof:

Assume that the derivative $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists.

we write $\Delta z = \Delta x + i\Delta y$

then $z + \Delta z = (x+iy) + (\Delta x + i\Delta y)$

$= x + \Delta x + i(y + \Delta y)$

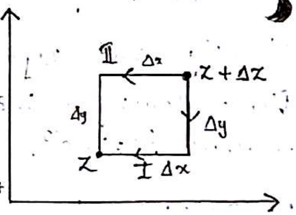
ie, $f'(z) = \lim_{\Delta z \to 0} \dfrac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - u(x,y) + i v(x,y)}{\Delta x + i \Delta y}$

$\left\{ z = x+iy,\ f(z) = u(x,y) + i v(x,y) \right\}$

first we get, $\Delta y \to 0$

and then ~~$\Delta x \to 0$~~, $\Delta z = \Delta x$

Now, when $\Delta y \to 0$



$= \lim_{\Delta x \to 0} \dfrac{u(x+\Delta x, y) + i\{V(x+\Delta x, y) - u(x,y) + i v(x,y)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \dfrac{V(x+\Delta x, y) - V(x,y)}{\Delta x}$

Since, $f'(z)$ exist, both the limits on the RHS exists ie, the partial derivatives of u and V with r.to x is exist.

ie, $f'(z) = u_x + i V_x$ .....(1)

Similarly, if we let $\Delta x \to 0$ first and then $\Delta y \to 0$, then $\Delta z \to i \Delta y$

then, $= \lim_{\Delta y \to 0} \dfrac{u(x, y+\Delta y) + i v(x, y+\Delta y) - u(x,y) + i v(x,y)}{i \Delta y}$

$= \lim_{\Delta y \to 0} \dfrac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + \cancel{i} \lim_{\Delta y \to 0} \dfrac{V(x, y+\Delta y) - V(x,y)}{\cancel{i} \Delta y}$

$\frac{1}{i} = \frac{i}{i^2} = \frac{-i}{1}$

$f'(z) = -i\, u_y + V_y$ ...... (2)

(1) = (2) $\Rightarrow$ $u_x + i V_x = -i u_y + V_y$

ie, $u_x = V_y$, $u_y = -V_x$

If $f'(z)$ exists, the four partial derivatives u_x, u_y, V_x, V_y exists.

By equating real and imaginary parts of $f'(z)$ in eqn (1) and (2), we get $u_x = V_y$, $u_y = -V_x$.

Hence, the Cauchy - Riemann eqns are satisfied.

$u_x = V_y$ in eqn (1) $\Rightarrow$ $f'(z) = V_y + i V_x$

$f'(z) = V_y + i V_x$

$f'(z) = u_x - i u_y$

$f'(z) = V_y - i u_y$.

★ Theorem: (Sufficient Condition):

If a real valued continuous function $u(x,y)$, and $V(x,y)$ of two real variables x & y have conti partial derivative that satisfies C-R equation in some domain D. Then the complex function $f(z) = u(x,y) + iV(x,y)$ is analytic in D.

? Show that the fu^n $f(z) = e^z$ is analytic everywhere. Also find its derivative ?

Ans. We have, $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x(\cos y + i\sin y)$

Here, $u(x,y) = e^x \cos y$

$V(x,y) = e^x \sin y$

$u_x = e^x \cos y$

$u_y = -e^x \sin y$

$V_x = e^x \sin y$

$V_y = e^x \cos y$

ie, $u_x = V_y$ and $V_x = -u_y$.

Hence, C-R equations are satisfied. Also the partial derivatives u_x, u_y, V_x, and V_y are continuous at all points. Therefore, $f(z) = e^z$ is analytic everywhere.

$f'(z) = u_x + iV_x = e^x \cos y + i e^x \sin y$

$= e^x(\cos y + i\sin y) = e^x \cdot e^{iy} = e^{x+iy} = e^z$

? Show that $f(z) = z^2$ is analytic and hence find its derivatives.

Ans. $f(z) = z^2 = x^2 - y^2 + 2ixy$

Here, $u(x,y) = x^2 - y^2$

$V(x,y) = 2xy$

$u_x = 2x$, $u_y = -2y$

$V_x = 2y$, $V_y = 2x$

$u_x = V_y$ and $u_y = -V_x$

Hence, Cauchy-R eqns are satisfied. Hence, Also u_x, u_y, V_x, and V_y are continuous.

$f'(z) = u_x + iV_x = 2x + i2y = 2(x+iy) = 2z$

? Show that $f(z) = \log z$ is differentiable except at $z=0$ and find its derivatives.

Ans. $f(z) = \log z = \log|z| + i\arg z$

$= \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$

$= \log(x^2+y^2)^{1/2} + i\tan^{-1}(y/x)$

$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

$u = \frac{1}{2}\log(x^2+y^2)$ & $V = \tan^{-1}(y/x)$

$u_x = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2} \times \cancel{2}x = \frac{x}{x^2+y^2}$

$u_y = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2} \times \cancel{2}y = \frac{y}{x^2+y^2}$

$V_x = \frac{1}{1+y^2/x^2} \times y \times \frac{-1}{x^2} = \frac{-y}{x^2(1+(y/x)^2)}$

$$= \frac{-y \times \cancel{x^2}}{\cancel{x^2}(x^2+y^2)} = -y/_{x^2+y^2}$$

$$V_y = 1/_{1+(y/x)^2} \times 1/_x = x/_{x^2+y^2}$$

Here, $U_x = V_y$ and $U_y = -V_x$. Hence, C-R eqns are satisfied and the function is analytic.

If $z=0$ then $x=0$ and $y=0$, then the partial derivative U_x, V_y, V_x, U_y doesn't exist. Hence, $\log z$ is not differentiable at $z=0$.

$$f'(z) = U_x + i V_x$$

$$= x/_{x^2+y^2} + i \; -y/_{x^2+y^2}$$

$$= x - iy/_{x^2+y^2} = \bar{z}/_{|z|^2}$$

$$= \bar{z}/_{z\bar{z}} = 1/_z$$

$$\therefore f'(z) = 1/_z$$

⇒ POLAR FORM OF C-R EQN:

If $f(z) = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$$= r\cos\theta + i r\sin\theta$$

$$f(z) = u(r,\theta) + i\, v(r,\theta)$$

then, the C-R equations are;

$$U_r = 1/_r V_\theta \quad \text{and} \quad V_r = -1/_r U_\theta$$

These are the C-R eqns in polar form.

For example;

$f(z) = z^2$

$$f(z) = z^2 = (re^{i\theta})^2 = r^2(e^{i\theta})^2 = r^2 e^{i2\theta}$$

$$f(z) = r^2[\cos 2\theta + i\sin 2\theta] = r^2\cos 2\theta + i r^2 \sin 2\theta$$

$$f(z) = u(r,\theta) + i\, v(r,\theta)$$

where, $u(r,\theta) = r^2\cos 2\theta$, $v(r,\theta) = r^2 \sin 2\theta$

Now, $U_r = \cos 2\theta \times 2r$; $U_\theta = r^2 \times -\sin 2\theta \times 2$

$V_r = \sin 2\theta \times 2r$; $V_\theta = r^2 \times \cos 2\theta \times 2$

ie, $U_r = \cos 2\theta \times 2r = 1/_r \times r^2 \cos 2\theta \times 2$

$$U_r = 1/_r \times V_\theta$$

Also, $V_r = \sin 2\theta \times 2r = -1/_r \times r^2 \times -\sin 2\theta \times 2$

$$V_r = -1/_r \times U_\theta$$

Hence, $U_r = 1/_r V_\theta$ and $V_r = -1/_r U_\theta$

ie, C-R equations are satisfied.

? Check whether C-R eqns are satisfied for the fn

$$f(z) = \cos\theta/_r - i \sin\theta/_r.$$

Ans. $f(z) = \cos\theta/_r - i\sin\theta/_r$

$U(r,\theta) = \cos\theta / r$, $V(r,\theta) = -\sin\theta / r$

Now; $U_r = \cos\theta \times -1/r^2$, $U_\theta = -1/r \sin\theta$

$V_r = -\sin\theta \times -1/r^2$, $V_\theta = -1/r \times \cos\theta$

Also, $1/r \times V_\theta = 1/r \times -1/r \times \cos\theta = -1/r^2 \cos\theta = U_r$

ie, $U_r = 1/r V_\theta$

And, $-1/r U_\theta = -1/r \times -1/r \sin\theta = 1/r^2 \sin\theta = V_r$

ie, $V_r = -1/r U_\theta$

Hence, C-R Equations are satisfied.

⇒ LAPLACE'S EQUATION & HARMONIC FUNCTION

If ϕ is a function of x and y, then the equation

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$$ is called laplace eqn.

or $\nabla^2\phi = 0$ or $\phi_{xx} + \phi_{yy} = 0$

Solutions of laplace eqn having continuous 2nd order partial derivatives are called harmonic functions.

eg: $\phi(x,y) = x^2 - y^2$

$\phi_x = 2x$, $\phi_{xx} = 2$

$\phi_y = -2y$, $\phi_{yy} = -2$

$\phi_{xx} + \phi_{yy} = 2 - 2 = 0$

Since, the function satisfies the laplace eqn, the function $\phi(x,y) = x^2 - y^2$ is harmonic fn.

⇒ THEOREM:

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then both u and v satisfies the laplace equation.

ie, $u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

In other words, the real and imaginary parts of an analytic function are harmonic.

? Verify whether the following functions are harmonic.

(i) $u = x^2 + y^2$

(ii) $u = xy$

(iii) $u = x^3 - 3xy^2$

(iv) $v = e^x \sin 2y$

Ans. (i) $u = x^2 + y^2$

$u_x = 2x$, $u_{xx} = 2$

$u_y = 2y$, $u_{yy} = 2$

ie, $u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$.

Hence, $u = x^2 + y^2$ is not a harmonic function

(ii) $u = xy$

$u_x = y$, $u_{xx} = 0$

$u_y = x$, $u_{yy} = 0$

ie $u_{xx} + u_{yy} = 0+0 = 0$

Hence, $u = xy$ is a harmonic function

(iii) $u = x^3 - 3xy^2$

$u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$

$u_y = -3x \times 2y = -6xy$, $u_{yy} = -6x$

$\therefore u_{xx} + u_{yy} = 6x - 6x = 0$

Hence, $u = x^3 - 3xy^2$ is a harmonic function.

(iv) $V = e^x \sin 2y$

$V_x = \sin 2y \times e^x$, $V_{xx} = \sin 2y \times e^x$

$V_y = e^x \cos 2y \times 2$, $V_{yy} = 2e^x \times -\sin 2y \times 2$

$V_{yy} = -4e^x \sin 2y$

ie, $V_{xx} + V_{yy} \neq 0$.

Hence, the fun $V = e^x \sin 2y$ is not harmonic.

HARMONIC CONJUGATE:

Th. u and V are real and imaginary part of an analytic function then V is said to be a harmonic conjugate of u.

$f = u + iV$

Note.

(i) V is a harmonic conjugate of u doesnot imply that u is harmonic conjugate of V.

eg: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 - i\,2xy$

$u = x^2 - y^2$, $V = 2xy$.

we know $f(z) = z^2$ is analytic, hence V is a harmonic conjugate of u.

but $V + iu = 2xy + i(x^2 - y^2)$ is nowhere analytic

Since, $V_x = 2y$

$u_y = -2y$

$V_x \neq u_y$

C-R eqn is not satisfied.

$\therefore 2xy + i(x^2 - y^2)$ is not analytic

(ii) If V is a harmonic conjugate of u in a domain D, then $-u$ is a harmonic conjugate of V in D.

(iii) If V is a harmonic conjugate of u, then $V + C$ where C is a constant is also harmonic conjugate of u.

x / mapping

Finding harmonic Conjugate of a function using C-R equation :

? Verify that $u = x^2 - y^2 - y$ is harmonic in the whole Complex plane and find a harmonic conjugate function $V(u)$.

Ans $u_x = 2x$, $u_{xx} = 2$,

$u_y = -2y - 1$, $u_{yy} = -2$

$u_{xx} + u_{yy} = 2 - 2 = 0$

hence, u is harmonic.

By C-R equations $u_x = V_y$

we have $u_x = 2x$ $\therefore V_y = 2x$ ... (1)

Integrating (1) w.r.t. y, we get

$V = \int 2x\,dy = 2xy + \phi(x)$ ....(2)

also from C-R eqns, $V_x = -u_y$

$V_x = -(-2y - 1)$

$V_x = 2y + 1$ ....(3)

Differentiating (2) w.r.t. x,

$V_x = 2y + \phi'(x)$ .... (4)

Comparing (3) & (4), we get

$2y + 1 = 2y + \phi'(x)$

$\phi'(x) = 1 \Rightarrow \phi(x) = x + c$

Sub. in eqn (2) $\Rightarrow V = 2xy + x + C$

$\therefore$ harmonic conjugate of $u = x^2 - y^2 - y$ is,

$V = 2xy + x + C$

? Find the harmonic Conjugate of $u = x^3 - 3xy^2 - 5y$.

Ans By C-R eqn, $u_x = V_y$, $V_x = -u_y$

$u_x = 3x^2 - 3y^2$, $V_y = 3x^2 - 3y^2$ ... (1)

$u_y = -5 - 6xy$, $V_x = -(-6xy - 5)$

$V_x = 6xy + 5$ ....(2)

Integrating (1) w.r.t. y,

$V = 3x^2y - 3y^3/3 + \phi(x)$

$V = 3x^2y - y^3 + \phi(x)$

$V_x = 6xy + \phi'(x)$ .... (3)

Comparing (2) and (3), we get $\phi'(x) = 5$

$\phi(x) = 5x + C$

$V = 3x^2y - y^3 + 5x + C$ //

doubt ? determine the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$

Ans. $u = e^{2x}(x\cos 2y - y\sin 2y)$

$f(z) = u + iv$

$u_x = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)(2e^{2x})$

$= e^{2x}\{\cos 2y + 2x\cos 2y - 2y\sin 2y\}$

$u_y = e^{2x}\{2x - \sin 2y \times 2 - (y \times \cos 2y + \sin 2y)\}$

$u_y = e^{2x}\{-2x\sin 2y - 2y\cos 2y - 2\sin 2y]$

$= -e^{2x}\{2x\sin 2y + 2y\cos 2y + 2\sin 2y\}$

By C.R equ, $V_y = u_x$

$V_y = e^{2x}\{\cos 2y + 2x\cos 2y - 2y\sin 2y\} \ldots (1)$

$V_x = -u_y$

$V_x = e^{2x}(2x\sin 2y + 2y\cos 2y + 2\sin 2y) \cdots (2)$

Integrating (1) w.r.t.y,

$V = e^{2x}\left[\frac{\sin 2y}{2} + 2x\frac{\sin 2y}{2} - 2\left\{\frac{-y\cos 2y}{2} + \frac{\sin 2y}{4}\right\}\right] + \phi(x)$

$= e^{2x}\left[\frac{\sin 2y}{2} + x\sin 2y + y\cos 2y - \frac{\sin 2y}{2}\right] + \phi(x).$

$V = e^{2x}[x\sin 2y + y\cos 2y] + \phi(x)$

$\int y\sin 2y = y\times\frac{-\cos 2y}{2} - 1\times$

$\int fg\,dx = f\int g(x)dx - \int f'(x)\int g(x)\,dx$

$\int fg\,dx = fg_1 - f'g_2 + f''g_3 \cdots$

$V_x = e^{2x}[\sin 2y] + [x\sin 2y + y\cos 2y]\times 2e^{2x} + \phi'(x)$

$= e^{2x}(\sin 2y + 2x\sin 2y + 2y\cos 2y) + \phi'(x) \ldots (3)$

Comparing (2) & (3), we get

$\phi'(x) = 0 \Rightarrow \phi(x) = C$, a constant

$\therefore V = e^{2x}(x\sin 2y + y\cos 2y) + C$

$f(z) = u + iv$

$= e^{2x}(x\cos 2y - y\sin 2y) + i[e^{2x}(x\sin 2y + y\cos$

$= e^{2x}x\cos 2y - e^{2x}y\sin 2y + ie^{2x}x\sin 2y +$
$ie^{2x}y\cos 2y + iC$

$f = e^{2x}x\cos 2y - e^{2x}y\sin 2y + ie^{2x}x\sin 2y + ie^{2x}$
$y\cos 2y + iC$

$= e^{2x}\cos 2y(x+iy) - e^{2x}\sin 2y(y - ix) + iC$

$= e^{2x}\cos 2y(x+iy) + ie^{2x}\sin 2y(x+iy) + C'$, where $C' = iC$

$= e^{2x}\cos 2y\, z + e^{2x}\sin 2y\, z$

$= ze^{2x}(\cos 2y + i\sin 2y) + C' = ze^{2x}e^{i2y} + C'$

$= z\cdot e^{2(x+iy)} + C' = ze^{2z} +$

Ans. 4)

★ TUTORIAL SESSION:

Problems:

doubt 1) Show that the function $f(z) = \bar{z}/z$ doesn't have a limit as $z \to 0$.

doubt 2) check whether $f(z) = z^2 - iz + 2$ is continuous at $z_0 = 1 - i$

3) find constants a, b, c for which $f(z) = (x + ay) + (bx + cy)$ is analytic.

4) find the value of 'a' if $u = x^3 + axy^2$ is harmonic

5) find the harmonic conjugate of $u = x^2 - y^2 - 2xy - 2x + 3y$.

Ans: 4) $u = x^3 + axy^2$

(2)

$u_x = 3x^2 + ay^2$

$u_{xx} = 6x + 0$, $u_y = 2axy$, $u_{yy} = 2ax$

$u_{xx} + u_{yy} = 0$

$6x + 2ax = 0$

$6x = -2ax \Rightarrow a = -3$ //

2) $f(z) = z^2 + iz + 2$

$f(z_0) = (1-i)^2 + -i(1-i) + 2$

LHS $= 1^2 - 2i + i^2 - i + i^2 + 2$

$= 1 - 2i - 1 - 1 - 1 + 2$

$= -3i + 1$

RHS $\lim_{z \to z_0} = f(z) = z^2 - iz + 2$

$= (1+i)^2 - i(1-i) + 2$

$= 1 - 3i$

here LHS = RHS. hence fun is differentiable & hence continuous.

$\lim_{z \to (1-i)} f(z) = f(z_0)$

$\therefore f(z)$ is continuous.

1) $f(z) = \bar{z}/z = x - iy / x + iy$

$\lim_{z \to 0} x - iy / x + iy = \lim_{x \to 0} \frac{x - imx}{x + imx} = \frac{1 - im}{1 + im}$

This limit depends on m,

so, when direction changes slope also change.

$\therefore$ this limit doesn't exist

5) $u = x^2 - y^2 - 2xy - 2x + 3y$

by CR equ, $u_x = v_y$, $v_x = -u_y$

(3)

$u_x = 2x - 2y - 2$

$v_y = 2x - 2y - 2 \ldots (1)$

$u_y = -2y + 2x + 3$

$v_x = 2y + 2x - 3 \ldots (2)$

Integrating (1) w.r.t. y,

$v = 2xy - 2y^2/2 - 2y = 2xy - y^2 - 2y + \phi(x)$

$v_x = 2y + \phi'(x) \ldots (3)$

Comparing (2) & (3),

$2y + \phi'(x) = 2y + 2x - 3$

$\phi'(x) = 2x - 3$

$\phi(x) = 2x^2/2 - 3x = x^2 - 3x + C$

$v = 2xy - y^2 - 2y + x^2 - 3x + C$ //

3) $f(z) = (x + ay) + i(bx + cy)$

$u(x,y) = x + ay$, $v(x,y) = bx + cy$

$u_x = 1$, $u_y = a$

$v_x = b$, $v_y = c$

$u_x = v_y \Rightarrow c = 1$

$v_x = -u_y \Rightarrow b = -a$